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# The Problem of Dirichlet for Quasilinear Elliptic Differential Equations with Many Independent Variables

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# THE PROBLEM OF DIRICHLET FOR QUASILINEAR ELLIPTIC DIFFERENTIAL EQUATIONS WITH MANY INDEPENDENT VARIABLES

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CONTENTS



This paper is concerned with the existence of solutions of the Dirichlet problem for quasilinear elliptic partial differential equations of second order, the conclusions being in the form of necessary conditions and sufficient conditions for this problem to be solvable in a given domain with arbitrarily assigned smooth boundary data. A central position in the discussion is played by the concept of global barrier functions and by certain fundamental invariants of the equation. With the help of these invariants we are able to distinguish an important class of 'regularly elliptic' equations which, as far as the Dirichlet problem is concerned, behave comparably to uniformly elliptic equations. For equations which are not regularly elliptic it is necessary to impose significant restrictions on the curvatures of the boundaries of the underlying domains in order for the Dirichlet problem to be generally solvable; the determination of the precise form of these restrictions constitutes a second primary aim of the paper. By maintaining a high level of generality throughout, we are able to treat as special examples the minimal surface equation, the equation for surfaces having prescribed mean curvature, and a number of other non-uniformly elliptic equations of classical interest.

## Introduction

Within recent years the study of the Dirichlet problem for second-order quasilinear elliptic equations has reached a certain degree of completeness, and, in particular, methods have been developed which reduce the existence question to one of obtaining *a priori* estimates for the solution and its gradient. I propose here to treat this latter problem, presenting sufficient detail to show the pre-eminent role which certain fundamental invariants of the equation play in the decision as to whether the Dirichlet problem is or is not generally solvable for a given equation and a given domain.

We consider, specifically, quasilinear elliptic partial differential equations of the form

$$\mathscr{A}(x, u, Du)\, D^2u = \mathscr{B}(x, u, Du). \tag{1}$$

Here $x = (x_1, \ldots, x_n)$ denotes points in the underlying $n$-dimensional Euclidean space, $Du$ and $D^2u$ denote respectively the gradient vector and Hessian matrix of the dependent variable $u = u(x)$, and $\mathscr{A}$ and $\mathscr{B}$ are respectively a given matrix and a given scalar function of the variables indicated. The multiplication convention implicit in (1) is the natural one which makes a scalar of the left-hand side; that is, using component notation,

$$\mathscr{A}\, D^2u = \mathscr{A}_{ij}\, \partial^2u/\partial x_i\, \partial x_j$$

where summation from 1 to $n$ over repeated indices is understood.

The structure of (1) is determined by the functions $\mathscr{A}(x, u, p)$ and $\mathscr{B}(x, u, p)$. We suppose throughout that they are defined and continuously differentiable for all real values of $u$ and $p$, and for all points $x$ in the closure of any particular domain under consideration. Ellipticity of (1) requires that the coefficient matrix satisfy the condition

$$\xi\mathscr{A}\xi > 0 \tag{2}$$

for arbitrary non-vanishing real vectors $\xi = (\xi_1, \ldots, \xi_n)$. Moreover, we can obviously assume without loss of generality that $\mathscr{A}$ is symmetric. Beyond this, any additional structure which will be required of the coefficient matrix and inhomogeneous term is best left to the statement of individual theorems. In any event, we shall maintain a fairly high level of generality throughout the paper, and accordingly will be able to treat as special examples the minimal surface equation, the equation for surfaces of prescribed mean curvature, and a number of other uniformly and non-uniformly elliptic equations of classical interest.

Roughly speaking, the conclusions are in the form of necessary conditions and sufficient conditions for the problem of Dirichlet in a given domain to be solvable for arbitrarily assigned smooth boundary data. The work thus constitutes a development and extension of the theory initiated by Bernstein in a series of papers from 1907 to 1912 and applied by him to a number of cases of great generality. Bernstein's work was subsequently sharpened and enlarged in an important and fundamental paper of Leray which appeared in 1939. In comparing this work with ours, we note first of all that their considerations were restricted to the case of two independent variables and involve various regularity assumptions on the behaviour of $\mathscr{A}$ and $\mathscr{B}$ for large values of $p$ which we avoid here. Bernstein had also concluded for certain equations (including that governing surfaces of constant mean curvature) that the Dirichlet problem was not generally solvable for arbitrary smooth data, even when the domain in question is convex and arbitrarily small. While strictly true, this statement nevertheless sidesteps the problem of determining precisely those domain for which the Dirichlet problem is in fact generally solvable. It is one of the goals of this paper (as it was of Leray's) to consider questions of just this sort.

In higher dimensions, the Dirichlet problem for non-uniformly elliptic equations has been comparatively less studied. We mention here specifically the work of Gilbarg and of Stampacchia, and recent papers of Trudinger and Oskolkov, and also a treatment by Jenkins & Serrin on the minimal surface equation. Some of the relations between this work and ours will be discussed below (see in particular §§ 9, 10 and 13).

The formal work of the paper is divided into four chapters. In the first one we give a precise statement of the problem under consideration and outline the general procedures to be followed in its study. The fundamental existence theorem upon which the further results of the paper are based is essentially due to Ladyzhenskaya & Uraltseva, though the form in which it has been stated does not seem to have appeared previously in the literature.

The second chapter is in many ways the central one of the paper; in it we treat in detail the various *a priori* estimates required by the existence theory. The results themselves are strongly influenced by the structure of the equation, and in particular by the behaviour of certain fundamental invariants. Following a discussion of maximum principles and global barrier functions, we introduce the concept of regular ellipticity. As far as the Dirichlet problem is concerned, the class of regularity elliptic equations behaves comparably to the class of uniformly elliptic equations. On the other hand, for equations which are not regularly elliptic it is frequently necessary to impose significant and important restrictions on the curvatures of the boundaries of the underlying domains in order to be able to solve the Dirichlet problem for arbitrarily given smooth data. An intimation of this state of affairs occurs already in the work of Bernstein and Leray, and in the paper by Jenkins & Serrin noted above. Bernstein had in fact seen the importance of convexity in the solution of Dirichlet's problem in two variables, while Leray had found a further relation between the second derivative of the boundary curve and the inhomogeneous term $\mathscr{B}$. For the minimal surface equation in higher dimensions, on the other hand, the mean curvature of the boundary of the underlying domain is the significant invariant. These various conditions are here seen as special cases of the criteria developed in §§ 9 and 10.

In §§ 12 and 13 we consider a number of interior estimates for the gradient vector, assuming that a boundary estimate is already available. This work is a clarification and elaboration of fundamental ideas due to Bernstein; we are in addition able to extend the results to a large class of equations in more than two variables, an extension which had not originally been thought possible in view of the inadequate number of critical point relations available when $n > 2$. Chapter II closes with a summary of various existence theorems which are direct consequences of the earlier *a priori* estimates.

Chapter III deals with certain cases of the non-solvability of Dirichlet's problem for arbitrary smooth data. It is shown that the results of chapter II are in many ways best possible, in that when their hypotheses fail to hold one may construct boundary data for which the Dirichlet problem cannot be solved. Similar results for two independent variables have been given by Bernstein, Leray, and Finn, and indeed our proof method is based on an interesting form of the maximum principle used by these authors.




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The paper is concluded with an extended discussion of various examples and special cases. These examples possess a considerable amount of independent interest, besides serving as concrete illustrations of the general theory. It should also be pointed out that, more often than not, the results of this chapter go beyond mere specialization of some earlier theorem of the paper. The following proposition may be singled out as indicative of the conclusions obtained (see also Serrin 1967):

*Let $\Omega$ be a bounded domain in n-dimensional Euclidean space, whose boundary is of class $C^2$. Then the Dirichlet problem in $\Omega$ for the equation of constant mean curvature*

$$[(1+|Du|^2)I - DuDu]\,D^2u = n\Lambda(1+|Du|^2)^{\frac{3}{2}}$$

*has a unique solution for arbitrary smooth boundary data, if and only if the mean curvature of the boundary surface is everywhere greater than or equal to $[n/(n-1)]\,|\Lambda|$.*

In the above example the notation $DuDu$ stands for the dyadic matrix generated by the gradient vector, while $I$ as usual denotes the identity matrix.

Particular importance attaches to this proposition in view of the geometric interest of the equation, the paucity of known exact solutions, and the number of conjectures which have been made concerning the Dirichlet problem in this case. Because of these circumstances I have included in the final part of the paper a fairly complete discussion of the background of this theorem, and have also stated generalizations for the case of prescribed (non-constant) mean curvatures and for surfaces with a single valued spherical projection. Similar theorems are also given for the closely related differential equation which describes the free surface of a stationary fluid under the combined action of gravity and surface tension.

Although the paper deals with both uniformly and non-uniformly elliptic equation on an equal footing, the former case has in fact received such careful and far-reaching study at the hands of other writers that our conclusions must be regarded for the most part as a contribution to the theory of non-uniformly elliptic equations. In this regard, it may be worthwhile to point out specifically the main theorems of the paper, as opposed to the various methods and techniques of *a priori* estimation used to attain them. In particular, we mention the five existence theorems of § 14, the non-existence theorems of §§ 16 and 18, and the four theorems of § 19 concerning the equation

$$\mathscr{A}(x, u, Du)\,D^2u = 0. \tag{1'}$$

As already mentioned, the examples discussed in chapter IV all have some special interest, either for their geometric significance or because they illustrate some particular point of the theory.

Theorems are numbered consecutively in each separate section. To avoid ambiguity in later references to a theorem, the section number is added as a distinguishing mark. Thus theorem 1 of § 8 is referred to in later sections as theorem 8·1.

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# Chapter I

In this chapter is given a precise statement of the Dirichlet problem and the general procedures to be followed in its study are outlined. The last section contains an important identity which is basic to a number of constructions later on in the paper.

## 1. The fundamental existence theorem

Let $\Omega$ be a bounded domain (connected open set) in $n$-dimensional Euclidean space. We shall use the notation $\partial\Omega$ to denote the boundary of $\Omega$ and $\overline{\Omega}$ to denote the closure of $\Omega$. The Dirichlet problem for equation (1) then consists in determining a function $u = u(x)$ such that:

(i) $u$ is twice continuously differentiable in $\Omega$ and continuous in $\overline{\Omega}$,

(ii) $u$ takes on given continuous boundary values $f$ on $\partial\Omega$, and

(iii) $\mathscr{L}u \equiv 0$ in $\Omega$, where the operator $\mathscr{L}$ is defined by

$$\mathscr{L}u \equiv \mathscr{A}(x, u, Du)\, D^2u - \mathscr{B}(x, u, Du).$$

In order to establish the existence of solutions of the Dirichlet problem one has available the classical technique of *a priori* estimation of the norms of eventual solutions. In particular, the following important proposition holds.

*Let the boundary of $\Omega$ and the assigned boundary values $f$ be of class $C^3$. Also let $\tau$ be an arbitrary real number in the closed interval $[0, 1]$. Suppose there exists a constant $M$, independent of $\tau$, such that the conditions*

(*i*) $v \in C^2(\overline{\Omega})$,

(*ii*) $v = \tau f$ *on* $\partial\Omega$,

(*iii*) $\mathscr{A}(x, v, Dv)\, D^2v = \tau\mathscr{B}(x, v, Dv)$ *in* $\Omega$

*imply* $\sup_\Omega (|v| + |Dv|) \leqslant M$. *Then the Dirichlet problem for equation* (1) *in* $\Omega$ *has at least one solution for the given boundary values* $f$. *Moreover, this solution is twice continuously differentiable in the closure of* $\Omega$.

The proof of this theorem involves two different sets of ideas—first, an *a priori* estimate concerning the equicontinuity of the gradient vector $Dv$, and second, a topological argument showing that solutions of (*i*), (*ii*), (*iii*) exist for each value of $\tau$, $0 \leqslant \tau \leqslant 1$. For two dimensions, such a programme was initiated and carried through by Bernstein under the assumption that the solutions $v$ are unique. Leray & Schauder (1934) then provided a general topological framework for the proof, avoiding Bernstein's uniqueness condition and at the same time giving a wide scope to the allowable homotopy. Moreover, the derivation of the requisite *a priori* estimates was greatly simplified by the work of Morrey and Nirenberg; cf. Nirenberg (1952). Thus the theorem for the case of two dimensions may be considered classical.

The topological apparature of Leray & Schauder applies unchanged in higher dimensions, but the *a priori* estimates concerning the equicontinuity of $Dv$ are now a much deeper matter. For the case of divergence structure equations these estimates were found by a number of authors on the basis of work of De Giorgi, Nash, and Moser, and corresponding existence theorems were stated by Ladyzhenskaya & Uraltseva (1961), Gilbarg (1963), and others. The first derivation of the requisite *a priori* estimates for the general case was given by Ladyzhenskaya & Uraltseva (1964). The above theorem follows as a consequence of their work and the Leray–Schauder theorem; for completeness we shall include the proof, using a method due to Gilbarg.

Let $w$ be an element of the Banach space $C^{1+\gamma}(\overline{\Omega})$, where $\gamma$ is a real number in the open interval $(0, 1)$ which will be fixed later. Consider the linear elliptic equation

$$a(x)\, D^2W = b(x) \tag{3}$$

where $$a(x) = \mathscr{A}(x, w(x), Dw(x)), \quad b(x) = \mathscr{B}(x, w(x), Dw(x)).$$

Since the coefficients $a(x)$ and $b(x)$ are Hölder continuous in the closure of $\Omega$, Schauder's theorem implies the existence of a unique solution $W \in C^{2+\gamma}(\overline{\Omega})$ which takes on the given boundary values $f$. This process clearly defines a mapping $T$ of $C^{1+\gamma}(\overline{\Omega})$ into itself, namely

$$w \to W = Tw.$$

Obviously $T$ maps bounded sets in $C^{1+\gamma}(\overline{\Omega})$ into bounded sets in $C^{2+\gamma}(\overline{\Omega})$, whence by Arzela's theorem and an easy argument (see the appendix at the end of this section) it follows that $T$ is completely continuous.

The proposition will now be proved if we can show that $T$ has a fixed point. To this end we apply the Leray–Schauder fixed point theorem in a simplified version due to Schaeffer. According to that result, $T$ will have a fixed point if there exists a constant $M'$, independent of $\tau$, such that

$$v = \tau T v \tag{4}$$

implies $\|v\| \leqslant M'$. Here $\tau$ denotes an arbitrary real number in $[0, 1]$, while the norm of course is that in the Banach space $C^{1+\gamma}(\overline{\Omega})$.

Let $v$ then be a solution of (4). By the definition of the mapping $T$ it is evident that $v = \tau f$ on $\partial\Omega$ and

$$\mathscr{A}(x, v, Dv)\, D^2v = \tau\mathscr{B}(x, v, Dv) \quad \text{in } \Omega. \tag{5}$$

Consequently, by the principal hypothesis of the proposition, we have

$$\sup_\Omega (|v| + |Dv|) \leqslant M. \tag{6}$$

To complete the proof it must further be shown that the quantity

$$\frac{|Dv(x) - Dv(y)|}{|x-y|^\gamma} \tag{7}$$

is uniformly bounded for all $x$, $y$ in $\Omega$ and all $\tau$ in $[0, 1]$.

In view of (6), the arguments $x$, $v$, $Dv$ in the coefficients of (5) are uniformly bounded. Consequently, as far as the solution $v$ is concerned, (5) may be considered uniformly elliptic and the functions $\mathscr{A}$ and $\tau\mathscr{B}$ uniformly of class $C^1$ in their arguments. Thus by an important theorem of Ladyzhenskaya & Uraltseva (1964), there exists a constant $\gamma$ (depending on $M$, on the structure of (1), an on norms for the boundary and the boundary data, but independent of $\tau$) such that the quantity (7) is uniformly bounded. Making this choice of $\gamma$ from the very beginning thereby completes the proof.

It is of interest to examine how the smoothness of the boundary and of the boundary data is used in the proof. To begin with, it is required in order to apply Schauder's theorem and thus to determine the function $W$; and, secondly, it is needed for certain boundary estimates in the theorem of Ladyzhenskaya & Uraltseva. Actually, Hölder continuity of the second derivatives of the boundary representations and the boundary data would suffice equally here, but this generalization would be pedantic. In fact, Trudinger (1967) has shown that a slightly modified version of the main proposition continues to hold without any conditions on the boundary data beyond continuity. Since Trudinger's modification requires further topological apparatus, and since the result as stated is both simple and adequate to our purposes, it will be unnecessary to consider these generalizations here.

*Remark.* The above result remains true if we replace conditions (*ii*) and (*iii*) by the more general conditions

(*ii'*) $v = k(\tau) f$ on $\partial\Omega$,

(*iii'*) $\mathscr{A}(x, v, Dv; \tau)\, D^2 v = \mathscr{B}(x, v, Dv; \tau)$ in $\Omega$,

where the functions $\mathscr{A}(x,u,p;\tau)$, $\mathscr{B}(x,u,p;\tau)$, and $k(\tau)$ are continuously differentiable in their arguments, satisfy $\xi\mathscr{A}\xi > 0$ for $\xi \neq 0$, and finally are such that

$$\mathscr{A}(x,u,p;1) = \mathscr{A}(x,u,p), \quad \mathscr{B}(x,u,p;1) = \mathscr{B}(x,u,p), \quad k(1) = 1$$

and
$$\mathscr{B}(x,u,p;0) = 0 \quad \text{for} \quad |p| \leqslant 1, \qquad k(0) = 0.$$

In order to prove this version of the result we consider the Dirichlet problem

$$\mathscr{A}(x,w,Dw;\tau)\, D^2 W = \mathscr{B}(x,w,Dw;\tau) + \mathscr{B}(x,w,Dw;0)\left(\frac{Dw \cdot DW}{|Dw|^2} - 1\right)$$

with $W = k(\tau) f$ on the boundary. This defines a mapping $T$ from $C^{1+\gamma}(\overline{\Omega}) \times [0,1]$ into $C^{1+\gamma}(\overline{\Omega})$, namely

$$(w,\tau) \to W = T(w,\tau),$$

and it is required to show that $T(\cdot, 1)$ has a fixed point. To this end we observe that $T$ is continuous and maps bounded sets into sequentially compact sets. Moreover, if $v = T(v,\tau)$ then $v$ satisfies (*i*), (*ii'*), and (*iii'*). Hence as in the proof already given, there exists a constant $M'$ independent of $\tau$ such that $\|v\| \leqslant M'$. Finally, $T = 0$ when $\tau = 0$. It follows therefore from the Leray–Schauder theorem (see Browder 1966) that $T$ has a fixed point when $\tau = 1$, completing the proof.

## Appendix

We shall show here that the mapping $Tw$ defined above is completely continuous. To begin with, we recall that $T$ maps bounded sets in $C^{1+\gamma}(\overline{\Omega})$ into bounded sets in $C^{2+\gamma}(\overline{\Omega})$. Hence by Arzela's theorem it is evident that $T$ maps bounded sets in $C^{1+\gamma}(\overline{\Omega})$ into sequentially compact sets in $C^{1+\gamma}(\overline{\Omega})$. It thus remains only to prove that $T$ is continuous.

Let $w \to w_0$ in $C^{1+\gamma}(\overline{\Omega})$. We must then show that $Tw \to Tw_0$ in the same space. Suppose for contradiction that this were not true. Since the elements $Tw$ are sequentially compact in $C^2(\overline{\Omega})$ as well as in $C^{1+\gamma}(\overline{\Omega})$, it follows that there is a sequence $\{w_m\}$ such that

$$w_m \to w_0 \quad \text{in} \quad C^{1+\gamma}(\overline{\Omega}), \quad W_m = Tw_m \to \hat{W} \quad \text{in} \quad C^2(\overline{\Omega})$$

where $\hat{W} \neq Tw_0$. On the other hand, by (3) we have

$$\mathscr{A}(x, w_m, Dw_m)\, D^2 W_m = \mathscr{B}(x, w_m, Dw_m)$$

whence letting $m$ tend to infinity there results

$$\mathscr{A}(x, w_0, Dw_0)\, D^2 \hat{W} = \mathscr{B}(x, w_0, Dw_0).$$

Consequently $\hat{W} = Tw_0$, and this contradiction completes the proof.

## 2. The existence programme

The proposition which was established in the previous section requires, for its application, the estimation of the $C^1$ norms of solutions of a family of Dirichlet problems. We propose to carry out this estimation in a series of steps which are described below and which are more or less unrelated to each other. If a particular equation possesses an appropriate structure so that all the various steps in this estimation process can be carried out, then the corresponding Dirichlet problem will have at least one solution.

The existence programme, then, will be based on the following succession of steps:

(A) The maximum absolute value of the solution is estimated in $\Omega$.

(B) Assuming the preceding step, one determines an estimate for the gradient of the solution at the boundary of $\Omega$.

(C) An estimate is then obtained for the gradient of the solution in the entire domain, depending on the results derived in the preceding steps.

Of course, it almost goes without saying that in this programme we are obliged to consider a *family* of Dirichlet problems and that the corresponding estimates must be independent of the homotopy parameter $\tau$. On the other hand, in presenting *a priori* estimates for the various steps of the programme it will not be necessary to retain the parameter $\tau$ in evidence, so long as the dependence of the estimates on the structure of the equation and on the given domain and data is set out in each case.

While there is some tendency for the existence programme as described to appear fragmentary, I think this need not be taken seriously: the subject itself is broad and simply does not in the nature of things possess a high degree of unity. In fact, looked at from another point of view, the requisite classification of non-linear structures in order to obtain meaningful results has its own particular interest.

The following chapter treats in detail the *a priori* estimates required by steps (A), (B), and (C) of the existence programme. Before turning to this, however, we shall derive an important identity which will be basic to a number of later constructions. This identity is contained in § 4; the next section contains certain preliminary material for the derivation.

## 3. The distance function

Consider an $(n-1)$-dimensional surface $S$ of class $C^3$ in $n$-dimensional Euclidean space, having no self-intersections and no boundary. We shall be interested in the function $d = d(x)$, defined as the distance from $x$ to the surface $S$. The following lemma holds.

Lemma 1. *Let the normal curvatures of $S$ be bounded in absolute value by $K$. Then $d$ is of class $C^2$ at all points whose distance from $S$ is less than $1/K$.*

*Proof.* For points $x$ whose distance from $S$ is less than $1/K$, we define $y = y(x)$ to be the (unique) point on $S$ nearest to $x$.

Now let $\bar{x}$ be a fixed point whose distance from $S$ is less than $1/K$. We consider a specific coordinate frame in which the $x_n$ axis is oriented along the normal to $S$ at $\bar{y} = y(\bar{x})$. In the neighbourhood of the point $\bar{y}$ the surface $S$ can then be represented in the form

$$x_n = \phi(x_1, \ldots, x_{n-1})$$

where $\phi$ is three times continuously differentiable. Moreover, for all points $x$ near $\bar{x}$ the following relations hold,

$$x_i = y_i - \frac{\phi_i(y)}{\sqrt{\{1+|D\phi(y)|^2\}}}d, \quad i = 1, \ldots, n-1; \quad x_n = \phi(y) + \frac{d}{\sqrt{\{1+|D\phi(y)|^2\}}}, \tag{8}$$

where $\phi_i = \partial\phi/\partial x_i$, $D\phi$ is the gradient vector of $\phi$, and $y = y(x)$. (There is a small point of notation here—the arguments of $\phi$, $\phi_i$, and $D\phi$ are, to be absolutely precise, $y_1, \ldots, y_{n-1}$.)

Relations (8) have the general form

$$x = f(y_1, \ldots, y_{n-1}, d) \tag{9}$$

where $f$ is a function of class $C^2$ in its arguments. The function $d(x)$, which can be considered as arising by inversion of (9), will then be of class $C^2$ at $\bar{x}$ provided the Jacobian of the transformation is non-zero at this point.

In order to compute this Jacobian it is clearly allowable to suppose that the coordinates $x_1, \ldots, x_{n-1}$ lie along the principal directions of $S$ at the point $\bar{y}$. In these coordinates the gradient $D\phi$ vanishes at $\bar{y}$ and the Hessian matrix $D^2\phi$ is diagonal there; consequently, since the diagonal elements are precisely the principal curvatures of $S$ at $\bar{y}$, an easy calculation yields the expression

$$\prod_1^{n-1} (1 - k_i d)$$

for the Jacobian at $\bar{x}$, where the numbers $k_i$ are the principal curvatures in question. This completes the proof of the lemma.

In what follows we shall require formulae for the first and second derivatives of $d$ at the point $x$. It will be sufficient to evaluate these in the special coordinate system used in the proof of lemma 1—that is, with the $x_n$ axis taken along the normal to $S$ at $\bar{y}$ and directed towards $\bar{x}$, and the remaining axes lying along the principal directions of $S$ at $\bar{y}$.

LEMMA 2. *In the special coordinate system introduced above, we have at* $\bar{x}$

$$Dd = (0, \ldots, 0, 1)$$

*and*

$$D^2 d = -\left(\frac{k_1}{1-k_1 d}, \ldots, \frac{k_{n-1}}{1-k_{n-1}d}, 0\right)_{\text{diagonal}}$$

*where* $k_1, \ldots, k_{n-1}$ *are the principal curvatures of* $S$ *at* $\bar{y}$.

*Proof.* Put $\xi = (y, \ldots, y_{n-1}, d)$ and write (8) in component form

$$x_i = f_i(\xi). \tag{10}$$

Differentiating (10) with respect to $x_j$ yields

$$\delta_{ij} = f_{i,l}\,\xi_{l,j} \tag{11}$$

where the Jacobian matrix $(f_{i,l})$ is diagonal at $\bar{x}$ with elements

$$(1-k_1 d, \ldots, 1-k_{n-1}d, 1).$$

Evaluating (11) at $\bar{x}$, it follows that $(\xi_{l,j})$ is also diagonal there, with elements

$$\left(\frac{1}{1-k_1 d}, \ldots, \frac{1}{1-k_{n-1}d}, 1\right).$$

The final row of this matrix is just $Dd$, proving the first part of the lemma.

If we now differentiate (11) with respect to $x_m$ there results

$$0 = f_{i,l}\,\xi_{l,jm} + f_{i,lk}\,\xi_{l,j}\,\xi_{k,m}$$

where $i, j, m = 1, \ldots, n$. Putting $i = n$ and evaluating this identity at $\bar{x}$ yields

$$0 = d_{,jm} + f_{n,jm}(1-k_j d)^{-1}(1-k_m d)^{-1} \tag{12}$$

(if either $j$ or $m$ equals $n$, the corresponding factor $1-k_j d$ or $1-k_m d$ should be suppressed). Now the matrix $(f_{n,jm})$ is easily found by direct calculation to be diagonal at $\bar{x}$ with elements

$$(k_1(1-k_1 d), \ldots, k_{n-1}(1-k_{n-1}d), 0).$$

Inserting this into (12), the lemma follows at once.

It is perhaps worth emphasizing that the principal curvatures $k_1, \ldots, k_{n-1}$ in lemma 2 are those arising when the unit normal to $S$ is directed *towards* the point $\bar{x}$.

## 4. A basic identity

Consider a bounded domain $\Omega$ whose boundary is of class $C^3$. Let $d = d(x)$ denote the distance from points $x$ in $\bar{\Omega}$ to the boundary surface. It follows from the results of the previous section that $d$ is twice continuously differentiable for $0 \leqslant d < d_0$, where $d_0 = 1/K$ and $K$ is an absolute bound for the normal curvatures of $\partial\Omega$.

For points $x$ in the boundary strip $0 \leqslant d \leqslant a$, we define

$$v(x) = g(x) + h(d)$$

where $g$ is twice continuously differentiable in $\bar{\Omega}$, and $h$ is continuous for $0 \leqslant d \leqslant a$ and twice continuously differentiable with respect to $d$ for $0 < d < a$. We suppose that

$$h' > 0 \quad \text{and} \quad a < d_0,$$

the latter condition guaranteeing that $v$ is of class $C^2$ in $0 < d < a$. Our purpose in what follows is to evaluate the expression

$$\mathscr{L}v = \mathscr{A}(x, v, Dv)\,D^2v - \mathscr{B}(x, v, Dv).$$

The result is conveniently expressed formally as

Lemma 1. *For $x$ in the boundary strip $0 < d < a$ we have*

$$\mathscr{L}v = \mathscr{F}h''/h'^2 - \mathscr{J}h' + \mathscr{A}D^2g - \mathscr{B}.$$

*The notation here has the following meaning: first,*

$$\mathscr{A} = \mathscr{A}(x, v, p), \quad \mathscr{B} = \mathscr{B}(x, v, p)$$

*where* $$p = p_0 + \nu h', \quad p_0 = Dg,$$

*and $\nu$ is the inner unit normal vector at the (unique) point $y(x)$ on $\partial\Omega$ nearest to $x$; and second,*

$$\mathscr{F} = (p-p_0)\mathscr{A}(p-p_0), \quad \mathscr{J} = \sum_1^{n-1} \frac{\lambda_i \mathscr{A} \lambda_i}{1-k_i d} k_i,$$

*where $k_1, \ldots, k_{n-1}$ and $\lambda_1, \ldots, \lambda_{n-1}$ are respectively the principal curvatures and principal directions of $\partial\Omega$ at $y(x)$.*

*Proof.* In order to evaluate $\mathscr{L}v$ at a particular point $\bar{x}$ in the boundary strip $0 < d < a$ it is convenient to introduce new coordinates (denoted with tildes) so that the $\tilde{x}_n$ axis coincides with the normal direction into $\Omega$ at $\bar{y} = y(\bar{x})$. We may further assume that the axes $\tilde{x}_1, \ldots, \tilde{x}_{n-1}$ are alined along the principal directions of the boundary surface at $\bar{y}$.

According to well known orthogonal transformation formulae, we have

$$\mathscr{L}v = \tilde{\mathscr{A}}(\tilde{x}, v, \tilde{D}v)\, \tilde{D}^2 v - \tilde{\mathscr{B}}(\tilde{x}, v, \tilde{D}v)$$

where $$\tilde{\mathscr{A}} = \mathscr{T}\mathscr{A}\mathscr{T}^{-1}, \quad \tilde{\mathscr{B}} = \mathscr{B},$$

and $\mathscr{T}$ is the orthogonal transformation relating the original and new coordinate systems, that is

$$\mathscr{T} = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_{n-1} \\ \nu \end{pmatrix}.$$

Using the definition of $v$ together with lemma 2 of the previous section, we find easily that

$$\tilde{D}^2 v = \tilde{D}^2 g - \left(\frac{k_1}{1-k_1 d} h', \ldots, \frac{k_{n-1}}{1-k_{n-1} d} h', -h''\right)_{\text{diagonal}} \quad \text{at } \bar{x}.$$

Consequently at $\bar{x}$, $$\mathscr{L}v = \tilde{\mathscr{A}}\tilde{D}^2 g - \tilde{\mathscr{B}} - \sum_1^{n-1} \frac{\tilde{\mathscr{A}}_{ii} k_i}{1-k_i d} h' + \tilde{\mathscr{A}}_{nn} h''. \tag{13}$$

Reverting to the original coordinates, one has

$$\tilde{\mathscr{A}}\tilde{D}^2 g = \mathscr{A} D^2 g, \quad \tilde{\mathscr{B}} = \mathscr{B},$$

and $$\tilde{\mathscr{A}}_{ii} = \lambda_i \mathscr{A} \lambda_i \quad (i = 1, \ldots, n-1), \quad \tilde{\mathscr{A}}_{nn} = \nu \mathscr{A} \nu.$$

Moreover, at $\bar{x}$ the arguments of $\mathscr{A}$ and $\mathscr{B}$ obviously are $\bar{x}$, $v$, and $Dv = Dg + h'Dd = p_0 + \nu h'$. By inserting these relations into (13), and noting that $\nu = (p-p_0)/h'$, the proof is completed.

A slightly different version of lemma 1 will also be useful. We formulate this as

Lemma 2. *For $x$ in the boundary strip $0 < d < a$ we have also*

$$\mathscr{L}v = \mathscr{F}\frac{h'' + Hh'}{h'^2} - \mathscr{K}h' + \mathscr{A}D^2 g - \mathscr{B}$$

*where $H$ denotes the mean curvature of $\partial\Omega$ at $y(x)$,*

$$\mathscr{K} = \sum_1^{n-1} \frac{\lambda_i \mathscr{A} \lambda_i}{1-k_i d} k_i + (\nu \mathscr{A} \nu)\, H,$$

*and the remaining notation is the same as in lemma 1.*

To obtain this result we simply add the quantity $\tilde{\mathscr{A}}_{nn} Hh'$ to the fourth term on the right-hand side of (13), and correspondingly subtract this quantity from the third term.





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The quantity $\mathscr{K}$ is a linear combination of the principal curvatures of the boundary surface. We note particularly the inequality

$$\mathscr{K} \geqslant k \operatorname{trace} \mathscr{A}$$

where $k$ stands for the minimum (signed) normal curvature of $\partial\Omega$ at the point $y(x)$. To prove this result, first observe that

$$\frac{k_i}{1-k_i d} \geqslant k_i \geqslant k, \quad H \geqslant k,$$

whence
$$\mathscr{K} \geqslant \left(\sum_1^{n-1} \lambda_i \mathscr{A} \lambda_i + \nu \mathscr{A} \nu\right) k.$$

The quantity in parentheses is simply the trace of $\mathscr{A}$ in the special coordinate frame $\lambda_1, \ldots, \lambda_{n-1}, \nu$. Since trace is an orthogonal invariant, the required inequality is proved.

By the same argument we find
$$\mathscr{J} \geqslant k \sum_1^{n-1} \lambda_i \mathscr{A} \lambda_i.$$

Since $k \geqslant -K$, where $K$ is an absolute bound for the curvatures of $\partial\Omega$, it follows easily that $\mathscr{J} \geqslant -K \operatorname{trace} \mathscr{A}$.



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## Chapter II

This chapter treats in detail the various *a priori* estimates required by steps (A), (B), and (C) of the existence programme described in § 2.

### 5. Basic definitions and agreements

We consider throughout this chapter a bounded domain $\Omega$ in $n$-dimensional Euclidean space, whose boundary $\partial\Omega$ is of class $C^3$. It will be assumed further that the boundary data under consideration is also of class $C^3$.

Under these assumptions, it is clear that there exist $C^3$ extensions of the boundary data into the closure of $\Omega$, while conversely any $C^3$ function defined in the closure of $\Omega$ generates $C^3$ boundary data by restriction. In consequence, *we may suppose without loss of generality that the boundary data in question is determined by a given function $f$ in $C^3(\overline{\Omega})$.*

We let

$$c_0 = \sup_\Omega |f|, \quad c_1 = \sup_\Omega |Df|, \quad c_2 = \sup_\Omega \|D^2 f\|,$$

where the norm $\|\mathscr{A}\|$ of a matrix is defined to be the root square of the sum of the squares of the components of $\mathscr{A}$. We also let $K$ be an upper bound for the unsigned normal curvatures of the boundary surface. Note that then the distance function $d = d(x)$ is of class $C^2$ in the boundary strip $0 \leqslant d < d_0$, where $d_0 = 1/K$.

For simplicity in what follows it will be convenient to normalize equation (1) by requiring

$$\operatorname{trace} \mathscr{A} = 1. \tag{14}$$

This condition will be tacitly understood in the formulation of all further results. As an immediate consequence of (2) and (14), the eigenvalues of $\mathscr{A}$ must lie between zero and one. Consequently we have

$$0 < \xi\mathscr{A}\xi < 1 \quad \text{for} \quad |\xi| = 1. \tag{15}$$

We shall be concerned throughout the paper with certain invariant functions, and it is convenient to place their definition at a central location. In particular, we set

$$\mathscr{C} = \mathscr{C}(x, u, p) = \mathscr{B}(x, u, p)/|p| \qquad (p \neq 0),$$

$$\mathscr{E} = \mathscr{E}(x, u, p) = p\mathscr{A}(x, u, p)\,p,$$

$$\mathscr{F} = \mathscr{F}(x, u, p, p_0) = (p - p_0)\,\mathscr{A}(x, u, p)\,(p - p_0).$$

It is clear that these expressions remain unchanged under orthogonal transformations of the underlying Euclidean space.

The invariant $\mathscr{E}$ was introduced in 1912 by Bernstein in his fundamental paper on the calculus of variations. In particular, Bernstein considered equations in two independent variables satisfying the condition

$$\mu_1 |p|^l \leqslant \mathscr{E} \leqslant \mu_2 |p|^l \quad (|p| \geqslant 1), \tag{16}$$

where $\mu_1$ and $\mu_2$ are positive constants and $l$ is a real number (actually Bernstein considered only integral values of $l$, but this restriction is unnecessary). It is evident that $l \leqslant 2$; the difference $g = 2 - l$ was called by Bernstein the *genre* of the equation.

We note, for example, that uniformly elliptic equations have genre zero, while the minimal surface equation on the other hand has genre two.† In the following work we shall not, however, restrict considerations solely to equations which have a well-defined genre.

Step (A) of the existence programme is discussed in § 6.

In §§ 7 through 11 we present several estimates for step (B), which apply under differing hypotheses concerning the structure of equation (1). Finally, in §§ 12 and 13 we treat step (C) of the existence programme. The last section of the chapter includes various existence theorems which are direct consequences of the earlier *a priori* estimates.

In what follows, a solution of (1) in $\Omega$ will always be understood to be a function $u$ in $C^2(\Omega)$ such that $\mathscr{L}u \equiv 0$ in $\Omega$. For convenience, we introduce one further notational convention: if any one of the variables $x$, $u$, or $p$ in some relation is not specifically quantified we assume that its range is, respectively, the closure of $\Omega$, the real numbers, or the real $n$-tuples.

## 6. Elementary maximum principles

We present in this section a number of maximum principles, all but one ultimately based on the well-known sign contradiction method at a maximum value of the dependent variable. While no claim is made for completeness, we have, nevertheless, attempted to give a useful selection of results, with the requirements of non-linear equations particularly in mind. Theorems 3, 4 and 6 seem to be new.

Theorem 1. *Let $\omega(x)$ be a function in $C^2(\Omega)$ such that*

$$\mathscr{L}(\omega+b) \leqslant 0 \quad \text{in} \quad \Omega$$

*for all constants $b > 0$. Suppose also that $u$ is a solution of* (1) *in $\Omega$ such that* $\limsup\,(u-\omega) \leqslant 0$ *as one approaches any point of the boundary. Then $u \leqslant \omega$ in $\Omega$.*

*Proof.* Let $v = u-\omega$ and $\kappa = \sup_\Omega v$. Suppose for contradiction that $\kappa$ is greater than zero. Then according to the boundary conditions there must exist a point $P$ in $\Omega$ where $v = \kappa$. If there is more than one such point we fix $P$ specifically so that any neighbourhood of $P$ contains points where $v < \kappa$; this is surely possible since $v < \kappa$ near the boundary.

Clearly there exists a neighbourhood $N$ of $P$ where

$$0 < v \leqslant \kappa, \quad v \not\equiv \kappa.$$

Now by hypothesis $$\mathscr{A}(x, \omega+b, D\omega)\, D^2\omega - \mathscr{B}(x, \omega+b, D\omega) \leqslant 0,$$

this inequality being valid for all $x$ in $\Omega$ and all positive real numbers $b$. Consequently it will hold for any positive function $b(x)$. Choosing $b = v$ in particular, and restricting consideration to the neighbourhood $N$ (hence $b > 0$), we obtain

$$\mathscr{A}(x, u, D\omega)\, D^2\omega - \mathscr{B}(x, u, D\omega) \leqslant 0 \quad \text{in} \quad N. \tag{17}$$

Moreover, since $u$ is a solution of (1),

$$\mathscr{A}(x, u, Du)\, D^2u - \mathscr{B}(x, u, Du) = 0 \quad \text{in} \quad N. \tag{18}$$

† It is easy to construct equations having any given non-negative genre $g$, for example,

$$[I + (1+|Du|^2)^{\frac{1}{2}g-1}(|Du|^2 I - DuDu)]D^2u = 0.$$

Subtracting (17) from (18) and applying the mean value theorem, it is easy to establish that

$$\mathscr{A}(x, u, Du)\, D^2 v + \text{linear function of } Dv \geqslant 0 \quad \text{in } N.$$

But then Hopf's maximum principle implies $v \equiv \kappa$ in $N$, which is a contradiction. This completes the proof of the theorem.

The next four theorems in this section are consequences of specific choices of the function $\omega(x)$.

THEOREM 2. *Suppose that*

$$u\mathscr{B}(x, u, 0) \geqslant 0 \quad \textit{for} \quad |u| > M.$$

*Let $u$ be a solution of* (1) *in* $\Omega$, *such that* lim sup $|u| \leqslant m$ *as one approaches any point of the boundary. Then* $|u| \leqslant \max(m, M)$ *in* $\Omega$.

*Proof.* Let $\omega(x) = \max(m, M)$. Then

$$\mathscr{L}(\omega + b) = -\mathscr{B}(x, \max(m, M) + b, 0) \leqslant 0.$$

Consequently, by theorem 1, $u \leqslant \omega$ in $\Omega$. Replacing $u$ with $-u$ we obtain a similar result, and the theorem is thereby proved.

THEOREM 3. *Let $\Omega$ be contained in a ball of radius $R$, and assume that*

$$\operatorname{sign} u\,.\,\mathscr{C}(x, u, p) \geqslant -1/R$$

*for* $|u| \geqslant M$ *and* $|p| \geqslant l(>0)$. *Let $u$ be a solution of* (1) *in* $\Omega$, *such that* lim sup $|u| \leqslant m$ *as one approaches points of the boundary. Then*

$$|u| < \max(m, M) + 2lR \quad \textit{in} \quad \Omega.$$

*Proof.* Let $O$ be the centre of the ball containing $\Omega$, and let $r$ denote distance from $O$. Consider the function

$$\omega(x) = \max(m, M) + lR(\mathrm{e} - \mathrm{e}^{r/R}), \quad 0 \leqslant r \leqslant R.$$

By direct calculation, $|D\omega| = l\,\mathrm{e}^{r/R}$ and

$$\mathscr{L}(\omega + b) = l\,\mathrm{e}^{r/R}\left\{\left(\frac{1}{r} - \frac{1}{R}\right)\frac{\vec{r}\mathscr{A}\vec{r}}{r^2} - \frac{1}{r} - \mathscr{C}\right\}$$

$$\leqslant l\,\mathrm{e}^{r/R}\left\{-\frac{1}{R} - \mathscr{C}\right\} \leqslant 0, \quad 0 < r \leqslant R,$$

where we have written $\vec{r}$ for the radius vector from $O$ and have used the main hypothesis at the last step (note that $\omega + b > M$ and $|D\omega| \geqslant l$).

We assert that $u \leqslant \omega$ in $\Omega$. Suppose for contradiction that this were not true. Then by theorem 1 and the fact that $u \leqslant \omega$ on the boundary of $\Omega$, the maximum of $u - \omega$ must occur at $r = 0$. This is impossible, however, in view of the functional form of $\omega$ at $r = 0$. Thus the assertion holds, and

$$u \leqslant \omega < \max(m, M) + 2lR \quad \text{in} \quad \Omega.$$

Replacing $u$ with $-u$ we obtain a similar result, and the theorem is proved.

COROLLARY. *Suppose that* $\mathscr{B} = o(|p|)$ *as* $|u|, |p| \to \infty$. *Let $u$ be a solution of* (1) *in* $\Omega$, *such that* lim sup $|u| \leqslant m$ *as one approaches points of the boundary. Then one can give an* a priori *estimate for* $|u|$ *in* $\Omega$.

The actual estimate depends on bounds for the order term. Since we shall not need this result, the details can be omitted.

THEOREM 4. *Suppose that for some fixed direction* $\nu$ *and for all* $\rho > 0$ *we have*

$$\frac{|\mathscr{B}(x,u,p)|}{\mathscr{E}(x,u,p)} \leqslant \phi(\rho), \quad p = \rho\nu,$$

*where* $\phi(\rho)$ *is a positive continuous function such that*

$$\int_0^\infty \frac{\mathrm{d}\rho}{\rho^2\phi(\rho)} = \infty.$$

*Let* $u$ *be a solution of* (1) *in* $\Omega$, *such that* $\limsup |u| \leqslant m$ *as one approaches any point of the boundary. Then* $|u| \leqslant L$ *in* $\Omega$, *where* $L$ *depends only on* $m$, $\phi$ *and the diameter of* $\Omega$.

*Proof.* Let $L$ denote the supporting plane of $\Omega$ which has normal direction $\nu$, and which is such that $\nu$ is directed toward the side of $L$ containing $\Omega$. Also let $d$ denote distance from $L$, and let $\bar{\delta}$ be the diameter of $\Omega$.

For any function $\omega(d)$ which is twice continuously differentiable and satisfies $\omega' > 0$ in the interval $0 \leqslant d \leqslant \bar{\delta}$, we have by an easy calculation

$$\mathscr{L}(\omega+b) = \mathscr{E}(x,\omega+b,p)\,\omega''/\omega'^2 - \mathscr{B}(x,\omega+b,p), \quad p = \omega'\nu.$$

Now let $\alpha$ and $\beta$ be positive numbers such that

$$\bar{\delta} = \int_\alpha^\beta \frac{\mathrm{d}\rho}{\rho^2\phi(\rho)}$$

and let $\omega$ and $d$ be parametrically related by the formulae

$$\omega = \int_\rho^\beta \frac{\mathrm{d}\rho}{\rho\phi(\rho)} + m, \quad d = \int_\rho^\beta \frac{\mathrm{d}\rho}{\rho^2\phi(\rho)}$$

where $\alpha \leqslant \rho \leqslant \beta$. We find easily that $\omega' = \rho$ and $\omega'' = -\omega'^2\phi(\omega')$. Consequently, using the main hypothesis of the theorem, $\mathscr{L}(\omega+b) \leqslant 0$. Therefore by theorem 1

$$u \leqslant \omega < m + \int_\alpha^\beta \frac{\mathrm{d}\rho}{\rho\phi(\rho)} \quad \text{in} \quad \Omega.$$

Since a similar result holds for $-u$, the proof is complete.

*Remarks.* A rather obvious generalization of theorem 4 can be given in which the main hypothesis is stipulated only for sufficiently large values of $|u|$, and also by using sign $u.\mathscr{B}(x,u,p)$ rather than $|\mathscr{B}(x,u,p)|$. We also note that a related result was proved by Ladyzhenskaya & Uraltseva (1961, page 79).

THEOREM 5. *Suppose that* $\mathscr{A}$ *is independent of* $u$ *and that* $\partial\mathscr{B}/\partial u \geqslant 0$. *Let* $\omega(x)$ *be a function in* $C^2(\Omega)$ *satisfying the condition*

$$\mathscr{L}\omega \leqslant 0 \quad in \quad \Omega.$$

*Suppose also that* $u$ *is a solution of* (1) *in* $\Omega$ *such that* $\limsup (u-\omega) \leqslant 0$ *as one approaches points of the boundary. Then* $u \leqslant \omega$ *in* $\Omega$.

This well known result is a direct consequence of theorem 1, for in view of the hypotheses on $\mathscr{A}$ and $\mathscr{B}$ we have

$$\mathscr{L}(\omega+b) \leqslant \mathscr{L}\omega \leqslant 0$$

for any positive constant $b$.

There is one further maximum principle rather similar to theorem 5, but applying when equation (1) has divergence structure. While this result will not be of immediate use, we shall nevertheless include the proof because of its special interest.

We suppose particularly that (1) can be written in the form

$$\partial\{A_i(x, u, Du)\}/\partial x_i = B(x, u, Du)$$

where $A_i$ and $B$ are respectively of class $C^2$ and $C^1$ in their arguments. We suppose further that the $(n+1)$ by $(n+1)$ matrix

$$\begin{pmatrix} \partial A_1/\partial p_1 & \dots & \partial A_1/\partial p_n & \partial A_1/\partial u \\ \vdots & & \vdots & \vdots \\ \partial A_n/\partial p_1 & \dots & \partial A_n/\partial p_n & \partial A_n/\partial u \\ \partial B/\partial p_1 & \dots & \partial B/\partial p_n & \partial B/\partial u \end{pmatrix} \tag{19}$$

is non-negative definite (here $(p_1, \dots, p_n)$ is the position variable for $Du$, as usual).

THEOREM 6. *Suppose that* (1) *has divergence structure and that the matrix* (19) *is non-negative definite.*

*Assume that* $u \in C^2(\bar{\Omega})$ *and* $v \in C^2(\bar{\Omega})$ *are solutions of* (1) *in* $\Omega$, *such that* $u-v \leqslant 0$ *on the boundary. Then* $u \leqslant v$ *in* $\Omega$.

*Proof.* Let $\kappa = \sup (u-v)$ in $\Omega$, and suppose for contradiction that $\kappa > 0$. For $\epsilon < \kappa$ put

$$w = \max(u-v-\epsilon, 0).$$

Clearly $w$ is Lipschitz continuous in $\Omega$ and vanishes in some neighbourhood of the boundary. Consequently a straightforward process of integration by parts yields

$$\int \{Dw \cdot A(x, u, Du) + wB(x, u, Du)\} \,\mathrm{d}x = 0$$

and

$$\int \{Dw \cdot A(x, v, Dv) + wB(x, v, Dv)\} \,\mathrm{d}x = 0,$$

and integrations taking place over the set where $w > 0$. If we now subtract these equations, there results

$$\int \{(Du-Dv) \cdot (A(x, u, Du) - A(x, v, Dv)) + (u-v)\ (B(x, u, Du) - B(x, v, Dv))\} \,\mathrm{d}x$$
$$-\epsilon \int \{B(x, u, Du) - B(x, v, Dv)\} \,\mathrm{d}x = 0.$$

Letting $\epsilon$ tend to zero then yields

$$\int \{(Du-Dv) \cdot (A(x, u, Du) - A(x, v, Dv)) + (u-v)\ (B(x, u, Du) - B(x, v, Dv))\} \,\mathrm{d}x = 0,$$

the integration now extending over the set where $u > v$.

The integrand is non-negative by the definiteness of the matrix (19), and consequently vanishes on the set where $u > v$. Now by a well-known application of the mean value theorem, the integrand can be written in the form

$$(Du-Dv)\frac{\partial \tilde{A}}{\partial p}(Du-Dv) + (Du-Dv) \cdot \left(\frac{\partial \tilde{A}}{\partial u} + \frac{\partial \tilde{B}}{\partial p}\right)(u-v) + \frac{\partial \tilde{B}}{\partial u}(u-v)^2,$$

where the notation has an obvious meaning and the tilde denotes evaluation at some intermediate value of the variables. The matrix $\partial A/\partial p$ is positive definite due to ellipticity, and moreover the intermediate value in question lies in a compact set of the $u, p$ space when $x$ is in $\overline{\Omega}$. Thus there exists a positive constant $\lambda$ such that

$$(Du-Dv)\,(\widetilde{\partial A/\partial p})\,(Du-Dv) \geqslant \lambda|Du-Dv|^2$$

for $x$ in $\overline{\Omega}$. Furthermore, by the Cauchy–Schwarz inequality

$$\begin{aligned}(Du-Dv)\cdot\left(\frac{\widetilde{\partial A}}{\partial u}+\frac{\widetilde{\partial B}}{\partial p}\right)(u-v) &\geqslant -\text{const.}|Du-Dv|\,.\,|u-v| \\ &\geqslant -\tfrac{1}{2}\lambda|Du-Dv|^2-\text{const.}(u-v)^2.\end{aligned}$$

Consequently we find $$|Du-Dv| \leqslant \text{const.}(u-v)$$

throughout the set where $u > v$.

This, however, is impossible. For let $P$ by any point where $u > v$ and let $Q$ be some nearest point to $P$ where $u = v$. (Such a point exists since $u \leqslant v$ at the boundary of $\Omega$.) By integrating the preceding differential inequality along the straight line from $Q$ to $P$ we then get

$$0 < u(P)-v(P) \leqslant [u(Q)-v(Q)]\exp(\text{const.}|P-Q|) = 0,$$

and the theorem is thereby proved.

Perhaps the most interesting application of theorem 6 is to first-order multiple integral variational problems in which the integrand $F(x, u, Du)$ is jointly convex in $u$ and $Du$ for each fixed value of $x$. For other maximum principles for elliptic equations, see Redheffer (1962).

## 7. Global barriers

Let $a$, $M$ be positive constants, and let $N$ denote the boundary neighbourhood $0 \leqslant d < a$. A *global barrier* is a function $v$ in $C^2(N)$ which satisfies the condition

$$\mathscr{L}(v+b) \leqslant 0 \quad \text{in} \quad N$$

for all positive constants $b$, and has the form

$$v(x) = f(x)+h(x),$$

where $h$ is continuous in the closure of $N$ and satisfies

$$h = 0 \quad \text{when} \quad d = 0, \qquad h = M \quad \text{when} \quad d = a.$$

A set of global barrier functions, defined for all $M > 0$, will be called a *global barrier family.*

The following lemma shows that when step (A) of the existence programme can be carried out, step (B) can be reduced to the problem of determining an appropriate global barrier.

Lemma. *Let $u \in C^2(\overline{\Omega})$ be a solution of the Dirichlet problem in $\Omega$. Suppose that $u \leqslant m$ and that there exists a global barrier function corresponding to $M = m+c_0$. Then*

$$\partial u/\partial n \leqslant L \quad on \quad \partial\Omega$$

*for every direction $\vec{n}$ into $\Omega$. Here the bound $L$ depends only on the global barrier function in question.*

*Proof.* Let $v = f+h$ be the global barrier function. Clearly $u = v$ on $\partial\Omega$. Also, recalling the definition of $c_0$ as the supremum of $|f|$ in $\Omega$, we have

$$u \leqslant v \quad \text{when} \quad d = a.$$

Consequently by theorem 6·1 $\qquad u \leqslant v \quad \text{in} \quad N.$

It now follows (since both $u$ and $v$ are differentiable on $\partial\Omega$) that

$$\frac{\partial u}{\partial n} \leqslant \frac{\partial v}{\partial n} \quad \text{on} \quad \partial\Omega$$

for every direction $\vec{n}$ into $\Omega$. Setting $L = \max_{\partial\Omega}|Dv|$ completes the proof.

## 8. Regularly elliptic equations

In this section we shall impose an appropriate degree of structure on the coefficient matrix $\mathscr{A}$ in order to carry out step (B) for an arbitrary domain $\Omega$ of class $C^3$. The definitions and agreements of § 5 will be maintained throughout.

We say that equation (1) is *m-regularly elliptic* provided that

$$\frac{1+|\mathscr{C}|}{\mathscr{E}} \leqslant \Phi(|p|) \quad \text{for} \quad |u| \leqslant m, \quad p \neq 0, \tag{20}$$

where $\Phi(\rho)$, $0 < \rho < \infty$, is a decreasing continuous function satisfying the condition

$$\int^{\infty} \frac{d\rho}{\rho^2\Phi(\rho)} = \infty. \tag{21}$$

Before proceeding further, it is worthwhile considering this definition in relation to several particular examples.

1. Suppose that $\mathscr{E} \geqslant \mu|p|$ for $|p| \geqslant 1$ and that for any $m > 0$

$$|\mathscr{B}|/\mathscr{E} \leqslant \gamma(m)\log(1+|p|) \quad \text{when} \quad |u| \leqslant m, \quad |p| \geqslant 1. \tag{22}$$

Then we can take $\qquad \Phi(\rho) = \dfrac{\mu^{-1}+\gamma\log(1+\rho)}{\rho}$

and (1) is regularly elliptic for each $m$.

If in particular equation (1) has a genre $g \leqslant 1$, then it is regularly elliptic provided that (22) holds. An even more special case occurs for uniformly elliptic equations, these being regularly elliptic provided that

$$|\mathscr{B}| \leqslant \gamma(m)\,|p|^2\log(1+|p|)$$

for $|u| \leqslant m$ and $|p| \geqslant 1$.

2. It may be worth presenting a specific example of a non-uniformly elliptic equation which is regularly elliptic, namely

$$(1+u_x^2)\,u_{xx}+2u_xu_yu_{xy}+(1+u_y^2)\,u_{yy} = 0$$

where $x$ and $y$ are independent variables. (The regular ellipticity is obvious since the genre is zero. On the other hand, the equation is not uniformly elliptic since its coefficient matrix has the same eigenvalues as the minimal surface equation.)

3. If an equation has genre $g > 1$ then it cannot be regularly elliptic, as one easily checks. In particular, neither the minimal surface equation nor the equation for surfaces of constant





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mean curvature are regularly elliptic. It is this feature which leads to the convexity and curvature restrictions which have proved necessary for the solution of the Dirichlet problem for these equations.

We may now turn to the main result of the section, after first proving a simple preliminary

LEMMA. *Suppose that* $|p_0| \leqslant c$. *Then*

$$\tfrac{1}{2}\mathscr{E} - 2c^2 \leqslant \mathscr{F} \leqslant 2\mathscr{E} + 2c^2.$$

*Proof.* Since $\mathscr{A}$ is positive definite and has unit trace, it is clear that for arbitrary real vectors $\xi$ and $\zeta$

$$|\xi\mathscr{A}\zeta| \leqslant \tfrac{1}{2}(\xi\mathscr{A}\xi + \zeta\mathscr{A}\zeta) \leqslant \tfrac{1}{2}(|\xi^2| + \zeta\mathscr{A}\zeta).$$

Setting $\xi = \mathscr{A}p$ and $\zeta = p$ then yields $|\mathscr{A}p|^2 \leqslant p\mathscr{A}p$. Now

$$\mathscr{F} = (p-p_0)\,\mathscr{A}(p-p_0) = p\mathscr{A}p - 2p_0\mathscr{A}p + p_0\mathscr{A}p_0.$$

For $|p_0| \leqslant c$ we have obviously $|p_0\mathscr{A}p| \leqslant c|\mathscr{A}p| \leqslant c\sqrt{(p\mathscr{A}p)}$. Hence, recalling the definition of $\mathscr{E}$, it follows that

$$\mathscr{E} - 2c\sqrt{\mathscr{E}} \leqslant \mathscr{F} \leqslant \mathscr{E} + 2c\sqrt{\mathscr{E}} + c^2.$$

The required conclusion is now an immediate consequence of Cauchy's inequality.

THEOREM 1. *Let* $u \in C^2(\overline{\Omega})$ *be a solution of the Dirichlet problem for equation* (1) *in* $\Omega$. *Assume that* $|u| \leqslant m$ *in* $\Omega$. *Then if* (1) *is m-regularly elliptic we have*

$$|Du| \leqslant L \quad \text{on} \quad \partial\Omega$$

*where L depends only on* $c_0$, $c_1$, $c_2$, $m$, $K$, *and the function* $\Phi$. ($K$, $c_0$, $c_1$, $c_2$ *are defined in* §5.)

*Proof.* We shall assume to begin with that (20) holds without the added restriction $|u| \leqslant m$; that is, we assume

$$\frac{1+|\mathscr{C}|}{\mathscr{E}} \leqslant \Phi(|p|) \tag{23}$$

for all $x$ in the closure of $\Omega$, all real $u$, and all $p \neq 0$. This assumption will be removed at the end of the proof.

Our goal is to construct an appropriate global barrier family and then to apply the lemma of the previous section. Let

$$v(x) = f(x) + h(d), \quad 0 \leqslant d \leqslant a,$$

where $a < d_0$. We assume that $h$ is twice continuously differentiable with respect to $d$ and that

$$h(0) = 0, \quad h(a) = M, \quad h'(d) \geqslant \alpha, \tag{24}$$

$\alpha$ being a positive constant whose value will be specified later. In order to evaluate $\mathscr{L}(v+b)$ we may apply lemma 4·1. Since $\mathscr{J} \geqslant -K$, this yields

$$\mathscr{L}(v+b) \leqslant \mathscr{F}h''/h'^2 + Kh' + \mathscr{A}D^2f - \mathscr{B}, \tag{25}$$

the arguments of $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{F}$ being $x$, $v+b$, and $p = p_0 + \nu h'$ where $p_0 = Df$.

Now $\mathscr{E} \geqslant 1/\Phi$, and $\Phi$ necessarily tends to zero as $|p|$ tends to infinity (since the integral (21) diverges). Consequently there exists a constant $\alpha_1$, depending only on $c_1$ and $\Phi$, such that

$$\mathscr{E} \geqslant 8c_1^2 \quad \text{for} \quad |p| \geqslant \alpha_1.$$

Now choose

$$\alpha = \max(c_1 + \alpha_1, MK, 1).$$

Since $p = p_0 + \nu h'$ and $|p_0| \leqslant c_1$ it is easy to check that

$$\alpha_1 \leqslant h' - c_1 \leqslant |p| \leqslant 2h'.$$

We are now in position to estimate the coefficients $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{F}$ in the inequality for $\mathscr{L}(v+b)$.

To begin with, since the eigenvalues of $\mathscr{A}$ all lie between 0 and 1, we have

$$|\mathscr{A}D^2f| \leqslant \|D^2f\| \leqslant c_2.$$

Next
$$|\mathscr{B}| \leqslant |p|\,|\mathscr{C}| \leqslant 2h'|\mathscr{C}|,$$

and finally, using the preliminary lemma and the fact that $\mathscr{E} \geqslant 8c_1^2$,

$$\mathscr{F} \geqslant \tfrac{1}{2}\mathscr{E} - 2c_1^2 \geqslant \tfrac{1}{4}\mathscr{E}.$$

Inserting these estimates into (25) yields

$$\begin{aligned}\mathscr{L}(v+b) &\leqslant \mathscr{F}h'\left\{\frac{h''}{h'^3} + 4\frac{c_2+K+2|\mathscr{C}|}{\mathscr{E}}\right\}\\ &\leqslant \mathscr{F}h'\left\{\frac{h''}{h'^3} + c\Phi(|p|)\right\} \quad (c = 4c_2+4K+8)\\ &\leqslant \mathscr{F}h'\left\{\frac{h''}{h'^3} + c\Phi(h'-c_1)\right\},\end{aligned}$$

the last step holding since $\Phi$ is a decreasing function of $|p|$.

We now choose $h'$ so that the expression in braces vanishes and so that conditions (24) hold. In order to do this, define $\beta$ by means of the relation

$$cM = \int_\alpha^\beta \frac{\mathrm{d}\rho}{\rho^2\Phi(\rho-c_1)}$$

(this is possible since the integral diverges). Next, let $h$ and $d$ be parametrically related by

$$ch = \int_\rho^\beta \frac{\mathrm{d}p}{\rho^2\Phi(\rho-c_1)}, \quad cd = \int_\rho^\beta \frac{\mathrm{d}p}{\rho^3\Phi(\rho-c_1)}$$

where $\alpha \leqslant \rho \leqslant \beta$. One easily checks that $h' = \rho \geqslant \alpha$ and that the quantity in braces vanishes. Moreover, it is evident that

$$h(0) = 0, \quad h(a) = M,$$

where
$$a = \frac{1}{c}\int_\alpha^\beta \frac{\mathrm{d}\rho}{\rho^3\Phi(\rho-c_1)},$$

(note that $a < M/\alpha \leqslant d_0$).

To summarize, we have constructed for each $M > 0$ a global barrier function with $a < d_0$. By the lemma of the previous section, therefore,

$$\partial u/\partial n \leqslant L \quad \text{on} \quad \partial\Omega$$

where $L$ depends only on the global barrier function corresponding to $M = m+c_0$. The latter depends, moreover, only on the quantities listed in the statement of the theorem. Replacing $u$ with $-u$ in the equation leaves the construction unaltered. Hence we also have

$$\partial u/\partial n \geqslant -L \quad \text{on} \quad \partial\Omega,$$

completing the proof of the theorem subject to the initial assumption (23).

If (23) does not hold, we consider a new equation with coefficient matrix $\hat{\mathscr{A}}$ defined by

$$\hat{\mathscr{A}}(x,u,p) = \begin{cases} \mathscr{A}(x,-m,p) & \text{if} \quad u < -m, \\ \mathscr{A}(x,u,p) & \text{if} \quad -m \leqslant u \leqslant m, \\ \mathscr{A}(x,m,p) & \text{if} \quad u > m, \end{cases}$$

and with a similarly defined inhomogeneous term $\hat{\mathscr{B}}$. Clearly $u$ is also a solution of the equation $\hat{\mathscr{A}}D^2u = \hat{\mathscr{B}}$, and this fact together with the evident relation

$$\frac{1+|\hat{\mathscr{C}}|}{\hat{\mathscr{E}}} \leqslant \Phi(|p|), \quad p \neq 0,$$

allows us to repeat the previous proof word for word, completing the demonstration of the theorem.

## 9. Boundary estimates depending on curvature

In this section we consider step (B) for equations which are not regularly elliptic. In such cases it turns out that the curvatures of the boundary surface must be restricted in order to obtain boundary estimates.

We shall consider the following conditions on the asymptotic behaviour of the invariants $\mathscr{A}$ and $\mathscr{C}$ for large values of $p$, namely

$$\mathscr{A}(x,u,p) = \mathscr{A}_0(x,\sigma)+o(1), \qquad \sigma = p/|p|, \tag{26}$$

and

$$\mathscr{C}(x,u,p) = \mathscr{C}_0(x,u,\sigma)+o(1), \quad \sigma = p/|p|, \tag{27}$$

as $p \to \infty$, where $\mathscr{A}_0(x,\sigma)$ and $\mathscr{C}_0(x,u,\sigma)$ are continuously differentiable functions of their arguments and where

$$\partial\mathscr{C}_0/\partial u \geqslant 0. \tag{28}$$

In these relations, as well as later in the paper, *order terms are assumed to be uniform for* $(x,u)$ *in any compact set, unless otherwise stated*. We emphasize that condition (28) will be tacitly assumed whenever we deal with the asymptotic relation (27).

The matrix $\mathscr{A}_0$ can be used to introduce a generalized mean curvature of the boundary surface. Let $y$ be a point on the boundary and let $\nu$ be the unit inner normal at $y$. Also let $k_1, \ldots, k_{n-1}$ and $\lambda_1, \ldots, \lambda_{n-1}$ be respectively the principal curvatures and principal directions of $\partial\Omega$ at $y$. We then put

$$A = A(y) = \mathscr{A}_0(y,\nu)$$

and define

$$\mathscr{H} = \mathscr{H}(y) = \sum_1^{n-1} (\lambda_i A\lambda_i)\, k_i + (\nu A\nu)\, H,$$

where $H$ is the ordinary mean curvature of $\partial\Omega$ at $y$. The function which arises upon replacing $\nu$ by $-\nu$ in this construction will be denoted by $\mathscr{I}(y)$.

It is worth emphasizing that $\mathscr{H}$ and $\mathscr{I}$ are *averages* of the principal curvatures† and that both quantities are orthogonal invariants, exactly as is the case with the mean curvature.

† That is, $\mathscr{H} = \sum_1^{n-1} a_i k_i$ where

$$a_i \geqslant 0, \quad \Sigma a_i = 1, \tag{*}$$

with a similar relation holding for $\mathscr{I}$. Conditions (*) are a direct consequence of two facts, (i) the numbers $(\lambda_i A\lambda_i)$ and $\nu A\nu$ are the diagonal elements of the matrix $A$ in the coordinate frame $\lambda_1, \ldots, \lambda_{n-1}, \nu$, and (ii) the matrix $\mathscr{A}_0$ necessarily has unit trace and is non-negative definite.

Moreover, if the matrix $A$ is proportional on $\partial\Omega$ to the Euclidean metric tensor, then $\mathscr{H} = \mathscr{I} = H$. This condition automatically holds in two dimensions, so that $\mathscr{H} = \mathscr{J} = \kappa$ in this case, where $\kappa$ is the curvature of the boundary curve.

To illustrate the constructions of $\mathscr{A}_0$, $\mathscr{C}_0$, $\mathscr{H}$ and $\mathscr{I}$, consider in particular the equation for surfaces of constant mean curvature in $n$ dimensions, namely

$$(1+|Du|^2)\,\Delta u - DuDu\,D^2u = n\Lambda(1+|Du|^2)^{\frac{3}{2}}.$$

After normalizing to unit trace, we find

$$\mathscr{A}_0 = \frac{I-\sigma\sigma}{n-1}, \quad \mathscr{C}_0 = \frac{n}{n-1}\Lambda;$$

moreover, the error terms in (26) and (27) are of degree $-2$. Evidently $\lambda_i A\lambda_i = (n-1)^{-1}$ and $\nu A\nu = 0$, and we have therefore

$$\mathscr{H} = \mathscr{I} = H$$

(this also follows from the fact that $A$ is proportional to the Euclidean metric on $\partial\Omega$).

THEOREM 1. *Let $u \in C^2(\bar{\Omega})$ be a solution of the Dirichlet problem for equation* (1) *in* $\Omega$. *Assume that $|u| \leqslant m$ in $\Omega$. If* (26) *and* (27) *hold, and if both conditions*

$$\mathscr{H} > -\mathscr{C}_0(y, f, \nu), \quad \mathscr{I} > \mathscr{C}_0(y, f, -\nu) \tag{29}$$

*are satisfied at each point $y$ of the boundary surface, then*

$$|Du| \leqslant L \quad \text{on} \quad \partial\Omega$$

*where $L$ depends only on $c_0$, $c_1$, $c_2$, $m$, $K$, a lower bound for the differences in* (29), *bounds for the error terms in* (26) *and* (27), *and $C^1$ norms of the functions $\mathscr{A}_0$ and $\mathscr{C}_0$.*

*Proof.* According to the argument used in the final step of the proof of theorem 8·1, we can assume without loss of generality that the error terms in (26) and (27), as well as the norms of $\mathscr{A}_0$ and $\mathscr{C}_0$, are uniformly bounded with respect to $u$. As before, we shall carry out the proof by constructing an appropriate global barrier family and applying the lemma of §7.

We put

$$v(x) = f(x) + h(d), \quad 0 \leqslant d \leqslant a$$

where $a < d_0$. It is further assumed that $h$ is twice continuously differentiable with respect to $d$ and that

$$h(0) = 0, \quad h(a) = M, \quad h'(d) \geqslant \alpha, \tag{30}$$

$\alpha$ being a positive constant which will be determined later. By lemma 4·2

$$\mathscr{L}(v+b) = \mathscr{F}\frac{h''+Hh'}{h'^2} - \mathscr{K}h' + \mathscr{A}D^2f - \mathscr{B}$$

the arguments of $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{F}$ being $x$, $v+b$, and $p = p_0 + \nu h'$.

Let $\theta > 0$ be chosen so that

$$\mathscr{H} \geqslant -\mathscr{C}_0(y, f, \nu) + 5\theta$$

at each point of the boundary surface (this can surely be done since the terms in (29) are continuous functions of $y$).

An easy estimate yields $|p| \geqslant \alpha - c_1$ and

$$\|\mathscr{A}(x, v+b, p) - \mathscr{A}_0(x, \nu)\| \leqslant \|\mathscr{A}(x, v+b, p) - \mathscr{A}_0(x, \sigma)\| + \frac{2c_1}{|p|}\sup\left\|\frac{\partial\mathscr{A}_0}{\partial\sigma}\right\|.$$

Moreover, if $y = y(x)$ denotes the point on the boundary nearest $x$, we have

$$\|\mathscr{A}_0(x,\nu)-\mathscr{A}_0(y,\nu)\| \leqslant d\sup\left\|\frac{\partial\mathscr{A}_0}{\partial x}\right\|.$$

Consequently, when $\alpha$ is suitably large and $d$ is sufficiently small, the difference

$$\mathscr{A}(x,v+b,p)-A(y)$$

becomes arbitrarily small. In particular, for $\alpha \geqslant \alpha_1$ and $d \leqslant a_1$,

$$\mathscr{K} \geqslant \sum_1^{n-1}(\lambda_i\mathscr{A}\lambda_i)\,k_i+(\nu\mathscr{A}\nu)\,H \geqslant \mathscr{H}-\theta. \tag{31}$$

Here $a_1$ and $\alpha_1$ depend only on $\theta$, $c_1$, $K$, bounds for the error term $\|\mathscr{A}-\mathscr{A}_0\|$, and a bound for the $C^1$ norm of $\mathscr{A}_0$.

Similarly, for $\alpha$ suitably large we have

$$\mathscr{C}(x,v+b,p) \geqslant \mathscr{C}_0(x,v+b,\nu)-\theta \geqslant \mathscr{C}_0(x,f(x),\nu)-\theta$$

since $\mathscr{C}_0$ is increasing in its second argument. Since $|Df| \leqslant c_1$ we may assume further that

$$\mathscr{C}_0(x,f(x),\nu) \geqslant \mathscr{C}_0(y,f(y),\nu)-\theta \quad \text{for} \quad d \leqslant a_2.$$

This leads to the required estimate

$$\begin{aligned}\mathscr{B}(x,v+b,p) &= |p|\,\mathscr{C}(x,v+b,p)\\ &\geqslant |p|\,\{\mathscr{C}_0(y,f,\nu)-2\theta\} \geqslant h'\{\mathscr{C}_0(y,f,\nu)-3\theta\}\end{aligned} \tag{32}$$

provided $\alpha \geqslant \alpha_2$ and $d \leqslant a_2$.

Now choose

$$\alpha = \max\,(\alpha_1, \alpha_2, M/a_1, M/a_2, MK, c_2/\theta).$$

Then according to (31) and (32) we have (provided $a < \min\,(a_1, a_2, d_0)$)

$$\mathscr{L}(v+b) \leqslant \mathscr{F}\frac{h''+Hh'}{h'^2}+h'\{-\mathscr{H}-\mathscr{C}_0(y,f,\nu)+5\theta\} \leqslant \mathscr{F}\frac{h''+Kh'}{h'^2},$$

where at the last line we have used the definition of $\theta$ and the fact that $H \leqslant K$.

We now define

$$h(d) = M\frac{1-e^{-Kd}}{1-e^{-Ka}}, \quad \text{where} \quad a = \frac{1}{K}\log\left(1+\frac{MK}{\alpha}\right).$$

It is easy to check that $h''+Kh' = 0$ and that conditions (30) hold. Moreover, $a < \min\,(a_1, a_2, d_0)$, so that finally

$$\mathscr{L}(v+b) \leqslant 0 \quad \text{for} \quad 0 < d < a.$$

The rest of the proof is almost exactly the same as that of theorem 8·1. The only significant change is that when $u$ is replaced by $-u$ we must simultaneously replace $\mathscr{A}(x,u,p)$ with $\mathscr{A}(x,-u,-p)$ and $\mathscr{B}(x,u,p)$ with $-\mathscr{B}(x,-u,-p)$. This accounts for the second condition in (29), in which $\mathscr{H}$ has been replaced by $\mathscr{I}$ and $\mathscr{C}_0(y,f,\nu)$ by $-\mathscr{C}_0(y,f,-\nu)$.

*Remark.* It is clear from the above proof that a result similar to theorem 1 can be obtained under a milder structure condition than (27), namely

$$\mathscr{C}_0^-(x,u,\sigma)-o(1) \leqslant \mathscr{C}(x,u,p) \leqslant \mathscr{C}_0^+(x,u,\sigma)+o(1).$$

In this case (29) should be replaced by the pair of inequalities

$$\mathscr{H} > -\mathscr{C}_0^+(y,f,\nu) \quad \text{and} \quad \mathscr{I} > \mathscr{C}_0^-(y,f,-\nu).$$

The rather more complicated nature of these conditions is the main reason for emphasizing the structure (27) and the condition (29).

Even with no structure imposed on the coefficient matrix, it is still possible to obtain a result of interest.

THEOREM 2. *Let $u \in C^2(\bar{\Omega})$ be a solution of the Dirichlet problem for equation* (1) *in* $\Omega$. *Assume that* $|u| \leqslant m$ *in* $\Omega$. *Then if* (27) *holds, and if the minimum normal curvature* $k$ *of the boundary surface satisfies the condition*

$$k > \mp\mathscr{C}_0(y,f,\pm\nu) \tag{33}$$

*at each point of* $y$ *of the boundary, we have*

$$|Du| \leqslant L \quad on \quad \partial\Omega$$

*where* $L$ *depends only on* $c_0$, $c_1$, $c_2$, $m$, $K$, *a lower bound for the differences in* (33), *a bound for the error term in* (27), *and the* $C^1$ *norm of the function* $\mathscr{C}_0$.

The proof is exactly the same as before, except that we use the relation $\mathscr{K} \geqslant k$ in place of the more difficult inequality (31).

*Comment*. Theorem 2 includes as a special case the boundary estimates used by Gilbarg and Stampacchia in their study of non-uniformly elliptic equations of the form

$$\partial\{A_i(Du)\}/\partial x_i = 0 \quad A_i \in C^2.$$

In fact, when this equation is written out we have $\mathscr{B} \equiv 0$, so that (33) becomes precisely the convexity condition imposed by these authors.

## 10. Boundedly non-linear equations

With a very slight strengthening of the asymptotic structure of $\mathscr{A}$ and $\mathscr{C}$, the equality sign can be allowed in conditions (29) and (33). Although at first glance there in no great gain in this, the fact that a number of important examples can be included here makes the stronger result well worth stating.

We say that equation (1) is $(c, m)$-*boundedly non-linear* provided that (26) and (27) hold and

$$\frac{\|\mathscr{A}-\mathscr{A}_0\| + |\mathscr{C}-\mathscr{C}_0| + |p|^{-1}}{\mathscr{F}} \leqslant \Phi(|p|) \tag{34}$$

for $|u| \leqslant m$, $|p_0| \leqslant c$, $|p| > c$, where $\Phi(\rho)$ is a decreasing continuous function satisfying the condition

$$\int^\infty \frac{d\rho}{1+\rho^2\Phi(\rho)} = \infty.$$

Let us consider some particular cases.

1. Suppose that $\mathscr{F} \geqslant \mu(m,c) > 0$ and

$$\frac{\|\mathscr{A}-\mathscr{A}_0\| + |\mathscr{C}-\mathscr{C}_0|}{\mathscr{F}} \leqslant \gamma(m,c)\,\frac{\log(1+|p|)}{|p|}$$

when $|p| \geqslant 1+c$. Then we can take

$$\Phi(\rho) = \frac{\mu^{-1}+\gamma\log(1+\rho)}{\rho}, \quad \rho \geqslant 1+c$$

and (1) is boundedly non-linear for each $m$ and $c$.

If equation (1) has a well-defined genre $g$, $1 < g < 2$, then it is boundedly non-linear provided

$$\|\mathscr{A}-\mathscr{A}_0\|+|\mathscr{C}-\mathscr{C}_0| \leqslant \gamma(m)\,|p|^{1-g} \quad \text{for} \quad |u| \leqslant m,\ |p| \geqslant 1$$

(the lemma of § 8 implies $\mathscr{F} \approx |p|^{2-g}$ for large $p$). Conversely, if $g > 2$ then (1) cannot be boundedly non-linear.

2. The case where (1) has genre two is particularly interesting in view of the large number of examples of this type, including the minimal surface equation and the equation for surfaces having constant mean curvature. The following lemma states a sufficient condition for an equation of genre two to be boundedly non-linear.

Lemma. *Assume that equation* (1) *has genre two and that*

$$\mathscr{A}-\mathscr{A}_0 = o(|p|^{-1}), \qquad \mathscr{C}-\mathscr{C}_0 = O(|p|^{-1})$$

*as $p$ tends to infinity. Then* (1) *is boundedly non-linear. If for large $p$ the least eigenvalue of $\mathscr{A}$ is bounded below by a positive multiple of $|p|^{-2}$, then we can replace $o(|p|^{-1})$ by $O(|p|^{-1})$ in the order condition for $\mathscr{A}-\mathscr{A}_0$.*

*Proof.* In view of the result of paragraph 1, it is enough to show that $\mathscr{F} \geqslant \mu$ for $|p| \geqslant 1+c$. Consider first the case where $\mathscr{A}-\mathscr{A}_0 = o(|p|^{-1})$. From (26) we have obviously

$$\mathscr{E} = [\sigma\mathscr{A}_0\sigma+o(1)]\,|p|^2.$$

Since $\mathscr{E} = O(1)$ by the genre assumption, it follows that $\sigma\mathscr{A}_0\sigma = 0$. Moreover, $\mathscr{A}_0$ must surely be non-negative definite. Consequently we have

$$\begin{aligned}\mathscr{F} = (p-p_0)\mathscr{A}(p-p_0) &\geqslant (p-p_0)(\mathscr{A}-\mathscr{A}_0)(p-p_0)\\ &= p(\mathscr{A}-\mathscr{A}_0)p+o(1)\\ &= p\mathscr{A}p+o(1) = \mathscr{E}+o(1).\end{aligned}$$

Now $\mathscr{E} \geqslant \mu_1$ for $|p| \geqslant 1$. Hence $\mathscr{F} \geqslant \frac{1}{2}\mu_1$ for all sufficiently large $p$, and the required conclusion follows at once from the fact that $\mathscr{F}$ is positive and continuous for $c < |p| < \infty$.

When the least eigenvalue of $\mathscr{A}$ is of the order $|p|^{-2}$, we have obviously

$$\mathscr{F} = (p-p_0)\mathscr{A}(p-p_0) \geqslant \text{const.}\frac{|p-p_0|^2}{|p|^2}$$

and the final part of the lemma is an immediate consequence of this relation.

3. We have remarked in the previous section that the error terms $\mathscr{A}-\mathscr{A}_0$ and $\mathscr{C}-\mathscr{C}_0$ for the minimal surface equation and the equation for surfaces of constant mean curvature are of degree $-2$. By the above lemma, therefore, these equations are boundedly non-linear.

We now turn to the main results of the section, which slightly extend the conclusions of theorems 9·1 and 9·2. Note that the expressions $\mathscr{H}$ and $\mathscr{I}$ were defined in § 9.

Theorem 1. *Let $u \in C^2(\Omega)$ be a solution of the Dirichlet problem for equation* (1) *in* $\Omega$. *Assume that $|u| \leqslant m$ in $\Omega$, and that* (1) *is* $(c_1, m)$-*boundedly non-linear. Then if both conditions*

$$\mathscr{H} \geqslant -\mathscr{C}_0(y,f,\nu), \quad \mathscr{I} \geqslant \mathscr{C}_0(y,f,-\nu) \tag{35}$$

*hold at each point $y$ of the boundary surface, we have*

$$|Du| \leqslant L \quad \text{on} \quad \partial\Omega$$

*where $L$ depends only on $c_0, c_1, c_2, m, K$, the function $\Phi(\rho)$, and $C^1$ norms of the functions $\mathscr{A}_0$ and $\mathscr{C}_0$.*

*Proof.* We may suppose without loss of generality that

$$\frac{\|\mathscr{A}-\mathscr{A}_0\|+|\mathscr{C}-\mathscr{C}_0|+|p|^{-1}}{\mathscr{F}} \leqslant \Phi(|p|) \quad (|p_0|\leqslant c_1,\ |p|>c_1).$$

We carry out the proof, as usual, by constructing an appropriate global barrier family and applying the lemma of § 7.

Choosing $v$ in the form $\quad v(x) = f(x)+h(d), \quad 0\leqslant d\leqslant a,$

where $a<d_0$ and where $h$ satisfies the conditions

$$h(0)=0, \quad h(a)=M, \quad h'(d)\geqslant\alpha, \tag{36}$$

we find by lemma 4·2

$$\mathscr{L}(v+b) = \mathscr{F}\frac{h''+Hh'}{h'^2}-\mathscr{K}h'+\mathscr{A}D^2f-\mathscr{B}$$

where the arguments of $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{F}$ are $x$, $v+b$, and $p=p_0+\nu h'$.

As in the proof of theorem 9·1,

$$\|\mathscr{A}(x,v+b,p)-A(y)\| \leqslant \|\mathscr{A}-\mathscr{A}_0\|+\frac{2c_1}{|p|}\sup\left\|\frac{\partial\mathscr{A}_0}{\partial\sigma}\right\|+d\sup\left\|\frac{\partial\mathscr{A}_0}{\partial x}\right\|.$$

It follows (see (31)) that

$$\mathscr{K}\geqslant\mathscr{H}-\sqrt{n}\,K\|\mathscr{A}-\mathscr{A}_0\|-\text{const.}(|p|^{-1}+d).$$

Similarly (see (32))

$$\begin{aligned}\mathscr{B}(x,v+b,p) &\geqslant |p|\{\mathscr{C}_0(y,f,\nu)-|\mathscr{C}-\mathscr{C}_0|-\text{const.}(|p|^{-1}+d)\}\\ &\geqslant h'\{\mathscr{C}_0(y,f,\nu)-2|\mathscr{C}-\mathscr{C}_0|-\text{const.}(|p|^{-1}+d)\}\end{aligned}$$

provided $\alpha\geqslant c_1$. (The constants in the last three lines are of course not the same, though all depend only on $c_1$, $K$, and the $C^1$ norms of the functions $\mathscr{A}_0$ and $\mathscr{C}_0$.)

Before inserting the preceding estimates into the identity for $\mathscr{L}(v+b)$, it is convenient to obtain a bound for $d$ in terms of $|p|$. To this end we assume $h''<0$, a condition which will be satisfied by the eventual choice of $h$. Then, for $\alpha\geqslant c_1$,

$$d\leqslant 2h'|p|^{-1}d\leqslant 2M|p|^{-1},$$

since by the convexity of $h$ we have $h'(d)d\leqslant h(d)-h(0)\leqslant M$.

With the help of the above estimates, it follows that

$$\begin{aligned}\mathscr{L}(v+b)\leqslant \mathscr{F}\frac{h''+Kh'}{h'^2}&+h'\{-\mathscr{H}-\mathscr{C}_0(y,f,\nu)+2c_2|p|^{-1}\\ &+\sqrt{n}\,K\|\mathscr{A}-\mathscr{A}_0\|+2|\mathscr{C}-\mathscr{C}_0|+\text{const.}(1+M)|p|^{-1}\}\end{aligned}$$

provided $\alpha\geqslant c_1$. Using the principal hypothesis (35) and setting

$$c=2+2c_2+\sqrt{n}\,K+\text{const.}(1+M)$$

yields finally

$$\begin{aligned}\mathscr{L}(v+b)&\leqslant h'\mathscr{F}\left\{\frac{h''}{h'^3}+c\left(\frac{1}{h'^2}+\Phi(|p|)\right)\right\},\\ &\leqslant h'\mathscr{F}\left\{\frac{h''}{h'^3}+c\left(\frac{1}{h'^2}+\Phi(h'-c_1)\right)\right\}\end{aligned}$$

since $\Phi$ is a decreasing function. (It should be noted here that the constant $c$ depends only on $c_1$, $c_2$, $K$, $M$, and $C^1$ norms of the functions $\mathscr{A}_0$ and $\mathscr{C}_0$.)

Now choose
$$\alpha = \max(c_1, MK)$$
and define $\beta$ by means of the relation
$$cM = \int_\alpha^\beta \frac{d\rho}{1+\rho^2\Phi(\rho-c_1)}$$
(this is always possible since the integral is divergent). The function $h(d)$ is finally determined parametrically by the formulae
$$ch = \int_\rho^\beta \frac{d\rho}{1+\rho^2\Phi(\rho-c_1)}, \quad cd = \int_\rho^\beta \frac{d\rho}{\rho[1+\rho^2\Phi(\rho-c_1)]}$$
where $\alpha \leqslant \rho \leqslant \beta$. One easily checks that $h' = \rho \geqslant \alpha$ and that (36) holds with
$$a = \frac{1}{c}\int_\alpha^\beta \frac{d\rho}{\rho[1+\rho^2\Phi(\rho-c_1)]}$$
(note that $a < M/\alpha \leqslant d_0$). Moreover
$$h'' = -ch'[1+h'^2\Phi(h'-c)]$$
so that $h'' < 0$ and $\mathscr{L}(v+b) \leqslant 0$.

The remaining part of the proof is by now standard, and consequently may be omitted.

THEOREM 2. *Let $u \in C^2(\overline{\Omega})$ be a solution of the Dirichlet problem for equation* (1) *in* $\Omega$. *Assume that $|u| \leqslant m$ in $\Omega$, and that* (34) *holds with the term $\|\mathscr{A}-\mathscr{A}_0\|$ omitted. Then if the minimum normal curvature $k$ of the boundary surface satisfies the condition*
$$k \geqslant \mp\mathscr{C}_0(y, f, \pm\nu) \tag{37}$$
*at each point $y$ of the boundary surface, we have*
$$|Du| \leqslant L \quad on \quad \partial\Omega$$
*where $L$ depends only on $c_0$, $c_1$, $c_2$, $m$, $K$, the function $\Phi(\rho)$, and the $C^1$ norm of the function $\mathscr{C}_0$.*

The proof is exactly the same as before, except that we use the relation $\mathscr{K} \geqslant k$ in place of the more difficult estimate for $\mathscr{K}$ in terms of $\mathscr{H}$.

The minimal surface equation is boundedly nonlinear, as we remarked earlier; moreover, in this case $\mathscr{H} = \mathscr{I} = H$, where $H$ is the mean curvature of the boundary surface. It is therefore evident from theorem 1 that step (B) in the existence programme can be carried out if
$$H \geqslant 0$$
at each point of the boundary surface. Now step (A) is trivial for the minimal surface equation, and step (C) follows from the fact that the derivatives of a solution satisfy the maximum principle. *Hence the Dirichlet problem for the minimal surface equation in a $C^3$ domain $\Omega$ is solvable for arbitrarily given $C^3$ boundary data provided that $H \geqslant 0$ at each boundary point.* This result was first proved by Howard Jenkins and the present author, and in fact holds equally for $C^2$ domains and continuous boundary data (see § 19).

## 11. Mixed equations

It may happen that (1) is regularly elliptic for certain directions of the vector $p$ and boundedly non-linear for other directions. As an example, consider the 'false' minimal surface equation
$$(1+u_y^2)\,u_{xx} + 2u_x u_y u_{xy} + (1+u_x^2)\,u_{yy} = 0 \tag{38}$$

where $x$ and $y$ are independent variables. Putting $p = (p_1, p_2)$ and recalling the normalization to unit trace, we find easily that

$$\mathscr{E} = \frac{|p|^2 + 4p_1^2 p_2^2}{|p|^2+2}.$$

If the vector $\sigma = p/|p|$ is bounded away from both coordinate directions, and $|p| \geqslant 1$, then

$$\mu_1 |p|^2 \leqslant \mathscr{E} \leqslant \mu_2 |p|^2$$

for appropriate constants $\mu_1$ and $\mu_2$. Thus (38) is regularly elliptic for these directions.

When $\sigma$ lies in a coordinate direction, on the other hand, then $\mathscr{E} \leqslant 1$ and the condition of regular ellipticity fails. The condition of bounded non-linearity holds, however, since (by an easy calculation)

$$\mathscr{F} \geqslant \frac{1}{2+(1+c)^2} \quad \text{for} \quad |p| \geqslant 1+c.$$

In the following theorem we present a fairly simple result indicating the type of conclusions which can be drawn in such mixed cases.

THEOREM. *Let $u \in C^2(\bar{\Omega})$ be a solution of the Dirichlet problem for equation* (1) *in* $\Omega$.

*Assume that $|u| \leqslant m$ in $\Omega$, that* (1) *is m-regularly elliptic for all values of $p$ whose directions $\sigma = p/|p|$ lie in some open set $R$ of the unit sphere, and finally that* (1) *is $(c_1, m)$-boundedly non-linear.*† *Then if*

$$\mathscr{H} \geqslant -\mathscr{C}_0(y, f, \nu), \quad \mathscr{I} \geqslant \mathscr{C}_0(y, f, -\nu)$$

*at each point $y$ of the boundary surface where $\nu \notin R$, we have*

$$|Du| \leqslant L \quad \textit{on} \quad \partial\Omega$$

*where $L$ depends only on $c_0$, $c_1$, $c_2$, $m$, $K$, the function $\Phi(\rho)$, and $C^1$ norms of the functions $\mathscr{A}_0$ and $\mathscr{C}_0$.*

The proof is a direct combination of the demonstrations of theorems 8·1 and 10·1. The assumption that $\Phi(\rho)$ is the same in both conditions (20) and (34) is used in order to construct a single function $h(d)$ satisfying the requirements of each proof separately.

Returning to equation (38), let $R$ be any fixed set $\{|\sigma_1| > \epsilon, |\sigma_2| > \epsilon\}$, where $\epsilon > 0$. Then

$$\frac{1}{\mathscr{E}} \leqslant \frac{\text{const.}}{|p|^2} \quad \text{for} \quad |p| \geqslant 1, \quad \sigma \in R$$

while clearly
$$\frac{\|\mathscr{A} - \mathscr{A}_0\| + |p|^{-1}}{\mathscr{F}} \leqslant \frac{\text{const.}}{|p|} \quad \text{for} \quad |p| \geqslant 1+c.$$

Choosing $\Phi(\rho) = \text{const.}/\rho$, $\rho \geqslant 1+c$, it is apparent that the hypotheses of the theorem are satisfied. Since steps (A) and (C) are trivial for the present case (for example, use theorem 6·2 and the fact that the first derivatives of a solution of (38) satisfy a maximum principle), we have therefore proved the following result:

*The Dirichlet problem for equation* (38) *in a $C^3$ domain $\Omega$ is solvable for arbitrarily given $C^3$ boundary data provided that the curvature $\kappa$ of the boundary curve is non-negative in the neighbourhood of each boundary point whose tangent lies along one of the coordinate directions.*

The condition that the boundary and the data be of class $C^3$ can be lightened to class $C^2$ by an approximation argument; we shall, however, omit the details.

† We suppose the function $\Phi(\rho)$ to be the same in both conditions (20) and (34); doubtless this assumption could be removed by a suitable modification of the proof.

It is interesting to contrast this result with the corresponding theorem for the minimal surface equation. The rather trivial reversal of the sign of the middle term in these equations, though leaving the eigenvalues of the coefficient matrix unaltered and apparently affecting the structure only slightly, in fact has a profound effect on the type of Dirichlet problem which can be solved.

## 12. Interior estimates for $|Du|$

In contrast to the work of preceding sections, the derivation of interior estimates for the gradient of a solution naturally divides into two cases, depending on whether $n = 2$ or $n > 2$. For $n = 2$ the results go back to Bernstein; we shall follow his ideas in outline, though not in detail, when treating that case.

We first introduce some notation. If $\mathscr{W}$ is a given scalar function of the variables $x$, $u$, $p$ we shall write

$$\mathscr{W}_x = (\partial\mathscr{W}/\partial x_1, \ldots, \partial\mathscr{W}/\partial x_n), \quad \mathscr{W}_u = \partial\mathscr{W}/\partial u, \quad \mathscr{W}_p = (\partial\mathscr{W}/\partial p_1, \ldots, \partial\mathscr{W}/\partial p_n).$$

Also for $p \neq 0$ we put $$\dot{\mathscr{W}} = \dot{\mathscr{W}}(x, u, p) = \sigma\cdot\mathscr{W}_x + |p|\mathscr{W}_u \tag{39}$$

where $\sigma = p/|p|$. When $\mathscr{W}$ is a vector or matrix function it is convenient to retain the same notation, with an obvious interpretation.

We shall require two further invariants of (1), namely

$$\mathscr{E}^* = \frac{\mathscr{E}}{|p|^2} \quad \text{and} \quad \mathscr{D} = \frac{\mathscr{C}}{1-\mathscr{E}^*}, \quad p \neq 0.$$

Note that $\mathscr{E}^* < 1$ since trace $\mathscr{A} = 1$. For the remainder of the section it will be supposed that $n = 2$. The following result then holds.

Theorem 1. *Suppose that* $$\dot{\mathscr{D}} + \mathscr{D}^2 \geq 0 \quad \textit{for} \quad |p| \geq L'.$$

*Let $u \in C^1(\bar{\Omega}) \cap C^3(\Omega)$ be a solution of (1) in $\Omega$, such that $|Du| \leq L$ on $\partial\Omega$. Then $|Du| \leq \max(L, L')$ in $\Omega$.*

*Proof.* Put $w = |Du|^2$ and let $\Omega'$ denote the open subset of $\Omega$ where $w \neq 0$. We claim that

$$\mathscr{A}D^2w = 2(1-\mathscr{E}^*)(\dot{\mathscr{D}} + \mathscr{D}^2)w + \mathscr{M}\cdot Dw, \qquad x \in \Omega'$$

where the arguments of $\mathscr{A}$, $\mathscr{D}$, and $\mathscr{E}^*$ are $x$, $u$, $Du$ and where $\mathscr{M} = \mathscr{M}(x)$ is a continuous function on $\Omega'$.

Before proving this identity, we note that the result of the theorem follows from it as an immediate consequence of the strong maximum principle.

In deriving the identity, it is convenient to use standard component notation for partial derivatives. Writing (1) in the form $$\mathscr{A}_{ij}u_{ij} = \mathscr{B}$$

and applying the operator $u_k(\partial/\partial x_k)$ to both sides, there results

$$\mathscr{A}_{ij}u_k u_{ijk} + |Du|\,\dot{\mathscr{A}}_{ij}u_{ij} + \frac{\partial\mathscr{A}_{ij}}{\partial p_l}u_k u_{kl} u_{ij} = |Du|\,\dot{\mathscr{B}} + \frac{\partial\mathscr{B}}{\partial p_l}u_k u_{kl}.$$

Now $w = u_k u_k$, whence $$w_i = 2u_k u_{ki}, \quad w_{ij} = 2u_k u_{kij} + 2u_{ki}u_{kj}. \tag{40}$$

Inserting these expressions into the preceding line gives

$$\tfrac{1}{2}\mathscr{A}_{ij}w_{ij} = \mathscr{A}_{ij}u_{ik}u_{jk} + |Du|\,(\dot{\mathscr{B}} - \dot{\mathscr{A}}_{ij}u_{ij}) + \tfrac{1}{2}w_l\left(\frac{\partial\mathscr{B}}{\partial p_l} - \frac{\partial\mathscr{A}_{ij}}{\partial p_l}u_{ij}\right). \tag{41}$$

Let us further temporarily introduce the classical $(p, q, r, s, t)$ notation for the partial derivatives of functions of two independent variables; that is, we put

$$p = u_1, \quad q = u_2, \quad r = u_{11}, \quad s = u_{12}, \quad t = u_{22}.$$

Then clearly

$$\left.\begin{aligned} pr+qs &= \tfrac{1}{2}w_1, \\ ps+qt &= \tfrac{1}{2}w_2, \\ Ar+2Bs+Ct &= D, \end{aligned}\right\} \tag{42}$$

the last line simply being equation (1) in the classical notation. Setting

$$E = Aq^2 - 2Bpq + Cp^2$$

and solving the linear system (42) for $r$, $s$, $t$, we find that

$$\begin{aligned} r &= \quad q^2D/E + \text{linear function of } Dw, \\ s &= -pqD/E + \text{linear function of } Dw, \\ t &= \quad p^2D/E + \text{linear function of } Dw, \end{aligned}$$

valid for $p^2+q^2 \neq 0$. From these results, it follows that

$$\dot{\mathscr{A}}_{ij}u_{ij} = D\dot{E}/E + \text{linear function of } Dw$$

and

$$\begin{aligned} \mathscr{A}_{ij}u_{ki}u_{kj} &= A(r^2+s^2) + 2B(rs+st) + C(s^2+t^2) \\ &= (r+t)\,(Ar+2Bs+Ct) + (A+C)\,(s^2-rt) \\ &= wD^2/E + \text{linear function of } Dw. \end{aligned}$$

Inserting the preceding expressions into (41) and rearranging terms proves the required identity (note here that $D = \mathscr{B}$ and $E = (p^2+q^2)\,(1-\mathscr{E}^*)$).

For the special case $\mathscr{A} = \mathscr{A}(p)$ the result of theorem 1 was obtained by Bernstein (1910, page 129; see also Leray 1939, page 255).

Theorem 1 gives an important method for dominating the gradient of a solution in terms of the gradient at the boundary. Bernstein noted (1912, pages 459–461) that the result could be further refined by a change of dependent variables, though in order to carry out the construction in this case it is also necessary to have an *a priori* estimate for the modulus of the solution itself. Because Bernstein's proof contains a serious inaccuracy (i.e. formula (7) on page 459 applies only if the coefficients of the second derivatives are independent of $(x, y, z)$, while on the other hand in the application of this formula on page 460 the coefficients of the transformed equation definitely involve the variable $z$), it is worthwhile presenting a complete discussion here. There are two essentially different situations to consider, roughly corresponding to the case when (1) is boundedly non-linear and when it is regularly elliptic. The results may be compared with those of Leray (1939, page 263).

Theorem 2. *Suppose that $n = 2$ and that for some constant $\mu > 0$ the invariant function $\mathscr{E}$ satisfies the conditions*

$$p\cdot\mathscr{E}_p \leqslant (1-\mu)\,\mathscr{E}, \quad \mathscr{E} \geqslant \mu \tag{43}$$

*at least for all sufficiently large values of* $p$, *say* $|p| \geqslant l$. *Assume that as* $p$ *tends to infinity*

$$\mathscr{E}_x, \mathscr{E}_u, \mathscr{E}_p = O(\mathscr{E}) \tag{44}$$

*and that* (27) *holds with the remainder* $\mathscr{R} = \mathscr{C} - \mathscr{C}_0$ *satisfying*

$$\mathscr{R}, \mathscr{R}_x, \mathscr{R}_u, |p|\,\mathscr{R}_p = O(\mathscr{E}/|p|). \tag{45}$$

*Let* $u \in C^1(\overline{\Omega}) \cap C^3(\Omega)$ *be a solution of* (1) *satisfying* $|u| \leqslant m$ *in* $\Omega$ *and* $|Du| \leqslant L$ *on* $\partial\Omega$. *Then*

$$|Du| \leqslant M \quad in \quad \Omega$$

*where* $M$ *depends only on* $\mu$, $m$, $l$, $L$, *bounds for the order terms listed above, and on the* $C^1$ *norm of the function* $\mathscr{C}_0$.

*Remarks*. Conditions of the type (43), (44), (45) are consistent with the assumption that equation (1) is boundedly non-linear, and indeed one of the main applications of theorem 2 will be to the equation for surfaces having prescribed mean curvature.

Although the hypotheses appear somewhat complicated at first sight, the reader will notice that they amount roughly to the statement that differentiation of the invariants $\mathscr{E}$ and $\mathscr{R}$ with respect to $x$ and $u$ does not alter their order, while differentiation with respect to $p$ operates as if the functions were rational. Thus conditions (44) and (45) are in fact 'natural' assumptions in the sense of Ladyzhenskaya & Uraltseva. This being the case, it is apparent that the crucial additional requirements for theorem 2 are the existence of an interior estimate for $|u|$ and the condition $p \cdot \mathscr{E}_p \leqslant (1-\mu)\,\mathscr{E}$.

*Proof of theorem* 2. Since $|u| \leqslant m$, we may assume without loss of generality that the order terms in (44) and (45), as well as the norm of the function $\mathscr{C}_0$, are bounded independently of $u$.

Now introduce a new dependent variable $\bar{u}$ by means of the transformation

$$u = \varphi(\bar{u}) \quad (\varphi'(\bar{u}) > 0).$$

One checks without difficulty that $\bar{u}$ satisfies the equation

$$\overline{\mathscr{A}}\, D^2\bar{u} = \overline{\mathscr{B}} \tag{46}$$

with $$\overline{\mathscr{A}} = \mathscr{A} \quad \text{and} \quad \overline{\mathscr{B}} = \varphi'^{-1}(\mathscr{B} + \omega\mathscr{E}),$$

where the arguments of $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{E}$ are $x$, $u$, $p = \varphi'(\bar{u})\bar{p}$ and where

$$\omega = -\varphi''/\varphi'^2.$$

In order to apply the result of theorem 1 to the new equation, we note that

$$\overline{\mathscr{C}} = \mathscr{C} + \omega\mathscr{E}/|p|, \quad \overline{\mathscr{E}}^* = \mathscr{E}^*.$$

Moreover, recalling that $p$ depends on $\bar{u}$, an easy calculation yields

$$\bar{\sigma} \cdot \frac{\partial}{\partial x} + |\bar{p}| \frac{\partial}{\partial \bar{u}} = \sigma \cdot \frac{\partial}{\partial x} + |p| \left( \frac{\partial}{\partial u} - \omega p \cdot \frac{\partial}{\partial p} \right).$$

Thus we find, with regard to equation (46),

$$\dot{\overline{\mathscr{D}}} + \overline{\mathscr{D}}^2 = \delta \left( \frac{\mathscr{C} + \omega\mathscr{E}/|p|}{1 - \mathscr{E}^*} \right) + \left( \frac{\mathscr{C} + \omega\mathscr{E}/|p|}{1 - \mathscr{E}^*} \right)^2,$$

where $\delta$ is an abbreviation for the differential operator written out in full on the preceding line. The expression $\dot{\overline{\mathscr{D}}}+\overline{\mathscr{D}}^2$ will be non-negative provided that

$$I \equiv (1-\mathscr{E}^*)\,\delta\left(\mathscr{C}+\frac{\omega}{|p|}\mathscr{E}\right)+\left(\mathscr{C}+\frac{\omega}{|p|}\mathscr{E}\right)\delta\mathscr{E}^* \geqslant 0. \tag{47}$$

We shall show that (47) holds for all sufficiently large values of $p$ when

$$\varphi(\bar{u}) = (1/a)\log\bar{u}$$

and $a$ is chosen to be a suitably large constant. For this choice of $\varphi$ it is easy to see that $\omega = a =$ constant.

Now $\delta\mathscr{E} = \sigma\cdot\mathscr{E}_x+|p|\,\mathscr{E}_u-a|p|p\cdot\mathscr{E}_p$, whence (44) and the first inequality of (43) imply

$$-(C+(1-\mu)\,a)\,|p|\,\mathscr{E} \leqslant \delta\mathscr{E} \leqslant C(1+a)\,|p|^2\mathscr{E} \tag{48}$$

for all $p$ sufficiently large, say $|p| \geqslant \alpha_1$; here we use the letter $C$ as a generic bound depending only on the order terms in (44) and (45) and the norm of the function $\mathscr{C}_0$. Similarly, from (45) and the second inequality of (43),

$$\delta\mathscr{C} = \delta\mathscr{C}_0+\delta\mathscr{R} \geqslant \sigma\cdot\frac{\partial\mathscr{C}_0}{\partial x}+\delta\mathscr{R} \geqslant -C(\mu^{-1}+1+a)\,\mathscr{E} \tag{49}$$

provided $|p| \geqslant \alpha_2$; here we have used the facts that $\partial\mathscr{C}_0/\partial u \geqslant 0$ and $p\cdot\partial\mathscr{C}_0/\partial p = 0$. From (48) and (49) we obtain

$$\delta(\mathscr{C}+a\mathscr{E}/|p|) \geqslant (\mu a^2-(1+2a+\mu^{-1})\,C)\,\mathscr{E}$$

and also, since $\delta\mathscr{E}^* = |p|^{-2}\,\delta\mathscr{E}+2a\mathscr{E}/|p|$,

$$((1+\mu)\,a-C)\,\frac{\mathscr{E}}{|p|} \leqslant \delta\mathscr{E}^* \leqslant \left(\frac{2a}{|p|}+(1+a)\,C\right)\mathscr{E}.$$

We now write $\mathscr{C}+a\mathscr{E}/|p| = \mathscr{C}_0+(\mathscr{R}+a\mathscr{E}/|p|)$ and estimate the two parts separately: thus

$$\mathscr{C}_0 \geqslant -C, \quad \mathscr{R}+\frac{a\mathscr{E}}{|p|} \geqslant (a-C)\,\frac{\mathscr{E}}{|p|}$$

for $|p| \geqslant \alpha_3$.

Let us assume a final choice of $a$ satisfying $a > C$. Using the estimates of the previous paragraph, the function $I$ defined by (47) then obeys

$$I \geqslant \mathscr{E}[(1-\mathscr{E}^*)\,(\mu a^2-(1+2a+\mu^{-1})\,C)+\mathscr{E}^*(a-C)\,((1+\mu)\,a-C)-(2aC+(1+a)\,C^2)]$$

provided $|p| \geqslant \max(\alpha_1, \alpha_2, \alpha_3, 1) \equiv L'$. Assuming $\mu < 1$, as is certainly allowable, it is easy to see that

$$I \geqslant \mathscr{E}[\mu a^2-(4C+C^2)\,a-(1+\mu^{-1})\,C-C^2].$$

Now choose $a$ (necessarily greater than $C$) so that the quantity in square brackets vanishes.

We have thus shown that $\dot{\overline{\mathscr{D}}}+\overline{\mathscr{D}}^2 \geqslant 0$ provided $|\bar{p}| \geqslant L'/\min\varphi'$. Therefore by theorem 1

$$|D\bar{u}| \leqslant \max(L, L')/\min\varphi'$$

and hence $$|Du| \leqslant \frac{\max\varphi'}{\min\varphi'}\max(L, L') \leqslant \max(L, L')\exp(2am),$$

completing the proof of theorem 2.

*Remark.* The hypothesis $\mathscr{E} \geqslant \mu$ was used during the proof only to estimate the term $\partial\mathscr{C}_0/\partial x$ in inequality (49). Consequently, this condition can be dropped from (43) if $\mathscr{C}_0$ is independent of $x$.

The following result is the analogue of theorem 2 for the case of regular ellipticity. Again the crucial assumption is an *a priori* estimate for $|u|$.

THEOREM 3. *Suppose that $n = 2$ and that for some constant $\mu > 0$ the invariant function $\mathscr{E}^*$ satisfies the condition*

$$\mathscr{E}^* \leqslant 1-\mu. \tag{50}$$

*Assume further that*

$$\mathscr{B}, \mathscr{B}_x, \mathscr{E}_x = O(\mathscr{E}) \tag{51}$$

*and*

$$\mathscr{B}_u, \mathscr{B}_p, \mathscr{E}_u, \mathscr{E}_p = O(\mathscr{E}/|p|). \tag{52}$$

*Let $u \in C^1(\overline{\Omega}) \cap C^3(\Omega)$ be a solution of (1) satisfying $|u| \leqslant m$ in $\Omega$ and $|Du| \leqslant L$ on $\partial\Omega$. Then*

$$|Du| \leqslant M \quad in \quad \Omega,$$

*where $M$ depends only on $\mu$, $m$, $L$, and bounds for the order terms listed above.*

*Proof.* As in the preceding proof we can assume without loss of generality that the order terms are bounded independently of $u$. Moreover, as in that proof, it will be enough to show that (47) holds for an appropriate choice of the function $\varphi(\overline{u})$ and for suitably large values of $p$.

Now assume $\omega > 0$. Then

$$\delta\mathscr{C} \geqslant -C(|p|^{-1}+\omega)\,\mathscr{E}, \quad |\delta\mathscr{E}| \leqslant C(1+\omega|p|)\,\mathscr{E}$$

for $|p| \geqslant \alpha_1$, where $C$ is a generic bound depending only on the order terms in (51) and (52). Consequently, since

$$\delta\omega = |p|\frac{\omega'}{\varphi'} \quad \text{and} \quad \delta|p|^N = -N\omega|p|^{N+1},$$

we have

$$\delta\left(\mathscr{C}+\frac{\omega}{|p|}\mathscr{E}\right) \geqslant \left(\frac{\omega'}{\varphi'}+\omega^2-(1+\omega)\,(|p|^{-1}+\omega)\,C\right)\mathscr{E}$$

and

$$|\delta\mathscr{E}^*| \leqslant (2\omega+(|p|^{-1}+\omega)\,C)\frac{\mathscr{E}}{|p|}.$$

Since also

$$\left|\mathscr{C}+\frac{\omega}{|p|}\mathscr{E}\right| \leqslant (C+\omega)\frac{\mathscr{E}}{|p|} \quad (|p| \geqslant \alpha_2)$$

we obtain the following estimate for $I$:

$$\begin{aligned} I &\geqslant \mathscr{E}[(1-\mathscr{E}^*)\,(\omega'/\varphi'+\omega^2-(1+\omega)\,(|p|^{-1}+\omega)\,C)-\mathscr{E}^*(C+\omega)\,(2\omega+(|p|^{-1}+\omega)\,C)] \\ &\geqslant \mu\mathscr{E}[\omega'/\varphi'-A(\omega^2+\omega+|p|^{-1})] \end{aligned}$$

where in the last step we have used (50) and put

$$A = (2+C)^2/\mu;$$

here $|p| \geqslant \max(\alpha_1, \alpha_2, 1)$.

We now choose $\varphi(\overline{u})$ according to the inverse relation

$$\overline{u} = \int_{-2m-1}^{\varphi-m-1} \frac{\mathrm{d}t}{(1-\mathrm{e}^{2At})^{1/2A}}. \tag{53}$$

Since $u$ varies in the range $(-m, m)$ the integral is clearly well defined and $\varphi'(\overline{u}) > 0$.

Setting $\psi = \varphi - m - 1$ one finds

$$\varphi' = (1 - e^{2A\psi})^{1/2A}, \quad \omega = (e^{-2A\psi} - 1)^{-1}$$

and

$$\omega'/\varphi' = 2A(\omega^2 + \omega).$$

Substituting this into the preceding inequality for $I$ yields then

$$I \geqslant A\mu\mathscr{E}(\omega^2 + \omega - |p|^{-1}) \geqslant A\mu\mathscr{E}(e^{2A\psi} - |p|^{-1}).$$

Therefore $I \geqslant 0$ provided

$$|p| \geqslant \max(\alpha_1, \alpha_2, 1, e^{2A(2m+1)}) \equiv L'.$$

The rest is similar to the proof of theorem 2. Indeed with the help of theorem 1 we find

$$|Du| \leqslant \frac{\max \varphi'}{\min \varphi'} \max(L, L') < 2\max(\alpha_1, \alpha_2, e^{4A(1+m)}, L)$$

and the demonstration of theorem 3 is completed.

*Comments*. By following the steps of the proof it is not hard to show that the theorem remains true if the order conditions on $\mathscr{B}_u$ and $\mathscr{E}_u$ are weakened to

$$\mathscr{B}_u \geqslant -o(\mathscr{E}), \quad \mathscr{E}_u = o(\mathscr{E}).$$

Indeed, with this change the final inequality for $I$ takes the form

$$I \geqslant A\mu\mathscr{E}(e^{2A\psi} - o(1));$$

thus again $I \geqslant 0$ for sufficiently large $p$, and the required conclusion follows immediately.

Note finally that condition (50) is satisfied for any equation with a well-defined genre $g > 0$, or, more generally, whenever the function $\mathscr{E}$ satisfies $\mathscr{E} \leqslant \mu_2 |p|^{2-\theta}$ for $\theta > 0$.

## 13. Interior estimates for $|Du|$ in higher dimensions

The estimates of the preceding section apply only in the case of two dimensions. While it would be highly desirable to have fully corresponding results for higher dimensions, we see no way to obtain such conclusions. Nevertheless, it is possible to state analogous theorems for certain special classes of equations.

In this section we consider equations for which the coefficient matrix has the representation

$$\mathscr{A}(x, u, p) = \mathscr{G}(x, u, p)\mathscr{A}'(p) + \mathscr{G}_1(x, u, p)pp, \tag{54}$$

where $\mathscr{G}$ and $\mathscr{G}_1$ are scalar functions and $\mathscr{A}'$ is a positive definite matrix. It can be assumed without loss of generality that

$$\operatorname{trace} \mathscr{A}' = 1, \tag{55}$$

if necessarily by multiplying $\mathscr{G}$ by an appropriate factor. Evidently $\mathscr{G} > 0$ since $\mathscr{A}$ is positive definite.

A large number of classical non-linear partial differential equations fall into the class of equations defined by (54). In particular, this representation is obviously available whenever $\mathscr{A}$ is independent of $x$ and $u$. The representation also holds for the Euler–Lagrange equation associated with regular variational problems of the form

$$\delta \int F(x, u, |Du|)\, dx = 0, \quad F \in C^3. \tag{56}$$

Indeed, a short calculation shows that in this case

$$\mathscr{A} = \frac{1}{1+G}\frac{I}{n} + \frac{G}{1+G}\sigma\sigma,$$

where $G = 2|p|^2 F''/nF'$ and the primes denote differentiation with respect to $|p|^2$. Thus

$$\mathscr{A}' = I/n, \quad \mathscr{G} = 1/(1+G), \quad \text{and} \quad \mathscr{G}_1 = G/(1+G)\,|p|^2$$

for the Euler–Lagrange equation in question.

Now putting

$$\mathscr{S} = \mathscr{C}/\mathscr{G}, \quad p \neq 0$$

and recalling the definition (39), we have the following result.

THEOREM 1. *Suppose that* (54) *holds and that*

$$\dot{\mathscr{S}} + \mathscr{S}^2 \geqslant 0 \quad \textit{for} \quad |p| \geqslant L'$$

*Let* $u \in C^1(\bar{\Omega}) \cap C^3(\Omega)$ *be a solution of* (1) *in* $\Omega$, *such that* $|Du| \leqslant L$ *on* $\partial\Omega$. *Then* $|Du| \leqslant \max(L, L')$ *in* $\Omega$.

*Proof*. Put $w = |Du|^2$ and let $\Omega'$ be the open subset of $\Omega$ where $w \neq 0$. We claim that

$$\mathscr{A}D^2 w \geqslant 2\mathscr{G}(\dot{\mathscr{S}} + \mathscr{S}^2)\,w + \mathscr{N}\cdot Dw, \quad x \in \Omega'.$$

where the arguments of $\mathscr{A}$, $\mathscr{G}$, and $\mathscr{S}$ are $x$, $u$, $Du$, and where $\mathscr{N} = \mathscr{N}(x)$ is a continuous function on $\Omega$.

To prove this inequality, we may assume without loss of generality that $\mathscr{G} = 1$ by dividing both sides of (1) by $\mathscr{G}$. Of course the normalization trace $\mathscr{A} = 1$ is then lost, but we shall not in fact require this for the proof.

With the new normalization in mind, we now apply the differential operator $u_k\,\partial/\partial x_k$ to both sides of (1), as in the proof of theorem 12·1. This yields easily

$$\mathscr{A}_{ij}u_k u_{ijk} + |Du|\,\dot{\mathscr{G}}_1 u_i u_j u_{ij} + \frac{\partial \mathscr{A}_{ij}}{\partial p_l}u_k u_{kl}u_{ij} = |Du|\,\dot{\mathscr{B}} + \frac{\partial \mathscr{B}}{\partial p_l}u_k u_{kl}.$$

Using (40) it follows that

$$\tfrac{1}{2}\mathscr{A}_{ij}w_{ij} = \mathscr{A}'_{ij}u_{ik}u_{jk} + |Du|\,\dot{\mathscr{B}} + \text{linear function of } Dw.$$

Now we have

$$(\mathscr{A}'_{ij}u_{ij})^2 \leqslant \mathscr{A}'_{ij}u_{ik}u_{jk} \tag{57}$$

(introduce new coordinates so that $(u_{ij})$ is diagonal; then

$$(\mathscr{A}'_{ij}u_{ij})^2 = (\Sigma\mathscr{A}'_{ii}u_{ii})^2 \leqslant (\Sigma\mathscr{A}'_{ii})\,(\Sigma\mathscr{A}'_{ii}u_{ii}^2) = \mathscr{A}'_{ij}u_{ik}u_{jk}$$

which proves (57) since both sides are invariant under orthogonal transformations). Furthermore

$$\begin{aligned}\mathscr{A}'_{ij}u_{ij} &= \mathscr{A}_{ij}u_{ij} + \text{linear function of } Dw \\ &= \mathscr{B} + \text{linear function of } Dw.\end{aligned} \tag{58}$$

Using (57) and (58) to eliminate $\mathscr{A}'_{ij}u_{ik}u_{jk}$ from the identity for $\mathscr{A}_{ij}w_{ij}$ then gives

$$\tfrac{1}{2}\mathscr{A}D^2 w \geqslant \mathscr{B}^2 + |Du|\,\dot{\mathscr{B}} + \text{linear function of } Dw, \tag{59}$$

and the required inequality follows at once.

The rest of the proof is an immediate consequence of Hopf's maximum principle, as in the case of theorem 12·1.

In order to obtain a result corresponding to theorem 2 of the preceding section it is necessary to specialize still further. We shall in fact suppose that (54) holds in the form

$$\mathscr{A}(x,u,p) = \mathscr{G}(x,u,p)\,\mathscr{A}'(\sigma) + \mathscr{G}_1(x,u,p)\,pp \tag{60}$$

where $\sigma = p/|p|$. Such a representation applies in particular to regular variational problems (56), as we have noted earlier.

THEOREM 2. *Assume* (60) *holds. Suppose* $n > 2$ *but otherwise let the hypotheses of theorem* 2 *of the preceding section be satisfied. Then the conclusion of that theorem also holds.*

*Proof.* We introduce the transformation

$$u = \varphi(\bar{u})$$

exactly as before. Then $\bar{u}$ obeys the equation $\bar{\mathscr{A}}D^2\bar{u} = \bar{\mathscr{B}}$ with

$$\bar{\mathscr{A}} = \mathscr{G}\mathscr{A}'(\bar{\sigma}) + \varphi'^2\mathscr{G}_1\bar{p}\bar{p} \quad \text{and} \quad \bar{\mathscr{B}} = \varphi'^{-1}(\mathscr{B}+\omega\mathscr{E}),$$

where the arguments of $\mathscr{G}$, $\mathscr{G}_1$, $\mathscr{B}$, and $\mathscr{E}$ are $x$, $u$, $p = \varphi'(\bar{u})\,\bar{p}$, and where

$$\omega = -\varphi''/\varphi'^2.$$

Evidently $\bar{\mathscr{A}}$ has the form (54), so that theorem 1 can be applied.

For this purpose, we observe that

$$\bar{\mathscr{S}} = \frac{\mathscr{C}+\omega\mathscr{E}/|p|}{\mathscr{G}}.$$

Now forming the trace of (60) there results

$$1 = \mathscr{G} + |p|^2\mathscr{G}_1$$

while also

$$\sigma\mathscr{A}\sigma = (\sigma\mathscr{A}'\sigma)\,\mathscr{G} + |p|^2\mathscr{G}_1.$$

Eliminating $|p|^2\mathscr{G}_1$ from the last two relations and solving for $\mathscr{G}$ yields

$$\mathscr{G} = \frac{1-\sigma\mathscr{A}\sigma}{1-\sigma\mathscr{A}'\sigma},$$

whence recalling the definition of the invariant $\mathscr{D}$, we find

$$\bar{\mathscr{S}} = (1-\sigma\mathscr{A}'\sigma)\frac{\mathscr{C}+\omega\mathscr{E}/|p|}{1-\sigma\mathscr{A}\sigma} = (1-\sigma\mathscr{A}'\sigma)\,\bar{\mathscr{D}}.$$

Thus, with regard to the transformed equation,

$$\dot{\bar{\mathscr{S}}} + \bar{\mathscr{S}}^2 = (1-\sigma\mathscr{A}'\sigma)\,\dot{\bar{\mathscr{D}}} + (1-\sigma\mathscr{A}'\sigma)^2\,\bar{\mathscr{D}}^2.$$

This quantity will be non-negative provided $\dot{\bar{\mathscr{D}}} \geqslant 0$, that is, provided the expression (47) in the proof of theorem 12·2 is non-negative. This being the case, the rest of the demonstration can be carried over word for word from the proof of theorem 12·2.

Theorem 2 applies particularly to boundedly non-linear equations whose coefficient matrix satisfies (60). For the case of regularly elliptic equations we have the following corresponding result.

THEOREM 3. *Assume* (60) *holds. Suppose* $n > 2$ *but otherwise let the hypotheses of theorem* 3 *of the preceding section be satisfied. Then the conclusion of that theorem also holds.*

This is proved in exactly the same way as theorem 2, except that in the final step we make use of the proof of theorem 12·3.

13′. Further estimates for $|Du|$

Here we present two additional interior estimates for the gradient of a solution. As was the case in §13, these also apply only to certain special equations.

Theorem 4. *Let $\lambda = \lambda(x, u, p)$ be the minimum eigenvalue of the coefficient matrix $\mathscr{A}$, and suppose*

$$\dot{\mathscr{C}} - \|\dot{\mathscr{A}}\|^2/4\lambda \geqslant 0 \quad \textit{for} \quad |p| \geqslant L'.$$

*Let $u \in C^1(\overline{\Omega}) \cap C^3(\Omega)$ be a solution of (1) in $\Omega$, such that $|Du| \leqslant L$ on $\partial\Omega$. Then $|Du| \leqslant \max(L, L')$ in $\Omega$.*

*Proof.* The identity (41) used in the proof of theorem 12·1 holds equally in any number of dimensions. Since

$$\mathscr{A}_{ij} u_{ik} u_{jk} \geqslant \lambda \|D^2u\|^2,$$

it follows that

$$\tfrac{1}{2}\mathscr{A} D^2 w \geqslant \lambda \|D^2u\|^2 + |Du| (\dot{\mathscr{B}} - \|\dot{\mathscr{A}}\| . \|D^2u\|) + \mathscr{P}(x) \cdot Dw$$

$$\geqslant (\dot{\mathscr{C}} - \|\dot{\mathscr{A}}\|^2/4\lambda) w + \mathscr{P}(x) \cdot Dw$$

using Cauchy's inequality, where $\mathscr{P} = \mathscr{P}(x)$ is continuous in $\Omega$. The conclusion of the theorem is now an immediate consequence of Hopf's maximum principle.

Theorem 5. *Suppose that*

$$\mathscr{A}_x = O(\lambda), \qquad \mathscr{A}_u, \mathscr{A}_p = O(\lambda/|p|)$$

*and*

$$\mathscr{B}, \mathscr{B}_x = O(\lambda |p|^2), \qquad \mathscr{B}_u, \mathscr{B}_p = O(\lambda |p|).$$

*Let $u \in C^1(\overline{\Omega}) \cap C^3(\Omega)$ be a solution of (1) satisfying $|u| \leqslant m$ in $\Omega$ and $|Du| \leqslant L$ on $\partial\Omega$. Then*

$$|Du| \leqslant M \quad \textit{in} \quad \Omega$$

*where $M$ depends only on $m$, $L$, and bounds for the order terms listed above.*

This result is essentially due to Ladyzhenskaya & Uraltseva (1961, page 45); we have retained in the formulation only those relations actually used by Ladyzhenskaya & Uraltseva in their proof, which accounts for the difference between their result and the one stated here. Since the proof of theorem 5 shows a close unity with previous results, it is worth including for completeness.

We make the usual change of dependent variable

$$u = \varphi(\bar{u})$$

and apply the result of theorem 4 to the transformed equation $\bar{\mathscr{A}} D^2\bar{u} = \bar{\mathscr{B}}$. Then as in the proof of theorem 12·3 we find

$$\dot{\bar{\mathscr{C}}} - \frac{1}{4\lambda}\|\dot{\bar{\mathscr{A}}}\|^2 = \delta\left(\mathscr{C} + \frac{\omega}{|p|}\mathscr{E}\right) - \frac{1}{4\lambda}\|\delta\mathscr{A}\|^2$$

$$\geqslant (\omega'/\varphi' - \omega^2)\mathscr{E} - (1+\omega)(1+\omega|p|) C\lambda |p| - (1+\omega^2|p|^2) C\lambda$$

$$\geqslant \lambda |p|^2 [\omega'/\varphi' - A(\omega^2 + \omega + |p|^{-1})] \qquad (|p| \geqslant \max(\alpha_1, 1))$$

where $A = 2C+1$ and $C$ is a generic bound depending only on the order terms in the theorem (it is assumed that $\omega'/\varphi' \geqslant \omega^2$, a condition which will in fact hold for the final choice of $\varphi$).

This much being shown, the rest of the argument can be carried over almost word for word from the proof of theorem 12·3. (In their proof Ladyzhenskaya & Uraltseva choose

$$\varphi'(\overline{u}) = \text{const.} \exp(-u^m)$$

rather than the function (53) used here; since both choices accomplish the same ultimate purpose it is a matter of taste which one is finally adopted.)

As in theorem 12·3, the order conditions on $\mathscr{A}_u$ and $\mathscr{B}_u$ can be weakened to

$$\mathscr{A}_u = o(\lambda), \quad \mathscr{B}_u \geqslant -o(\lambda |p|^2)$$

without affecting the conclusion.

The parallel developments of theorems 1 to 3 of § 12, theorems 1 to 3 of § 13, and theorems 4 and 5 of the present section, has probably not escaped the reader's notice. A further connexion arises from the fact that the following well-known result can be considered as a consequence of either theorem 1 of § 12, theorem 1 of § 13, or theorem 4 of the present section.

*Suppose that* $\mathscr{A} = \mathscr{A}(p)$ *and* $\mathscr{B} = \mathscr{B}(u,p)$ *where* $\partial\mathscr{B}/\partial u \geqslant 0$. *Let* $u \in C^1(\overline{\Omega}) \cap C^3(\Omega)$ *be a solution of* (1) *in* $\Omega$, *such that* $|Du| \leqslant L$ *on* $\partial\Omega$. *Then* $|Du| \leqslant L$ *in* $\Omega$.

Indeed, using theorem 12·1 and supposing $n = 2$ we have

$$\dot{\mathscr{D}} + \mathscr{D}^2 = (1-\mathscr{E}^*)^{-1}\mathscr{B}_u + \mathscr{D}^2 \geqslant 0;$$

using theorem 13·1 we have $\quad \dot{\mathscr{S}} + \mathscr{S}^2 = \mathscr{B}_u + \mathscr{S}^2 \geqslant 0;$

and using theorem 4 of the present section

$$\dot{\mathscr{C}} - \|\dot{\mathscr{A}}\|^2/4\lambda = |p|^{-1}\mathscr{B}_u \geqslant 0.$$

Still another proof of this result may be given by using theorem 6·5 together with a consideration of the difference quotients of $u$.

*Remarks*. Other estimates for $|Du|$ have been given by Hartman & Stampacchia (1966) and by Trudinger in his Stanford dissertation. These authors considered the case when equation (1) can be written in divergence form,

$$\frac{\partial}{\partial x_i}\{A_i(x,u,Du)\} = B(x,u,Du).$$

Since we shall not require these results in the sequel, the detailed conclusions can be omitted here. On the other hand, it may be remarked that the condition $\mathscr{C} = O(\lambda)$ is an important hypothesis in this work. Consequently the results apply to non-uniformly elliptic equations only when $\mathscr{C}_0 = 0$, a feature which evidently restricts their usefulness and indicates the need for theorems of the type derived in the two preceding sections.

Further estimates for the gradient of solutions of divergence form equations have been given by Oskolkov (1967), though again the structure which he treats has the property that $\mathscr{C}_0 = 0$ whenever the equation is not regularly elliptic.

## 14. Summary

In chapter I we outlined a general existence programme for the Dirichlet problem, based upon a series of *a priori* estimates. According to the discussion of § 2, if a particular equation possesses appropriate structure so that one can carry out the estimates required at each step of the programme, then the corresponding Dirichlet problem will have at least one solution.

In the present section we shall illustrate this theory by giving a representative selection of existence theorems based directly on the *a priori* estimates developed earlier in the chapter. We retain the notations and conventions of § 5, except that in the interest of generality the theorems are formulated for boundaries and boundary data which are only of class $C^2$. The first result applies to a class of equations which are regularly elliptic.

THEOREM 1. *Let* $\Omega$ *be a bounded domain in n-dimensional Euclidean space whose boundary is of class* $C^2$. *Assume either* $n = 2$ *or that* (60) *holds.*

*Suppose there exists a constant* $\mu > 0$ *such that the invariant function* $\mathscr{E} = p\mathscr{A}p$ *satisfies the condition*

$$\mu|p| \leqslant \mathscr{E} \leqslant (1-\mu)\,|p|^2$$

*at least for all sufficiently large values of p. Assume further that as p tends to infinity*

$$\mathscr{B},\, \mathscr{B}_x,\, \mathscr{E}_x = O(\mathscr{E})$$

*and*

$$\mathscr{B}_u,\, \mathscr{B}_p,\, \mathscr{E}_u,\, \mathscr{E}_p = O(\mathscr{E}/|p|),$$

*and finally suppose* $\mathscr{B}$ *and* $\mathscr{E}$ *satisfy the hypotheses of either theorem* 6·2 *or* 6·4.

*Then the Dirichlet problem for equation* (1) *in* $\Omega$ *is solvable for arbitrarily given boundary data of class* $C^2$.

*Proof.* We assume first that the domain and the boundary data are of class $C^3$, as in § 5, and that the functions $\mathscr{A}(x, u, p)$ and $\mathscr{B}(x, u, p)$ are of class $C^2$. These restrictions will be removed at the end of the proof.

In order to apply the fundamental theorem (§ 1), consider a function $v \in C^2(\overline{\Omega})$ such that

$$\mathscr{A}(x, v, Dv)\, D^2v = \tau\mathscr{B}(x, v, Dv) \quad \text{in} \quad \Omega \tag{61}$$

and $v = \tau f$ on $\partial\Omega$.

Evidently $|v| \leqslant c_0$ on $\partial\Omega$; therefore by theorem 6·2 or 6·4, whichever may be applicable, it is clear that

$$|v| \leqslant m' \quad \text{in} \quad \Omega$$

where $m'$ is a constant independent of $\tau$. Moreover,

$$\frac{1+\tau|\mathscr{C}|}{\mathscr{E}} = O(|p|^{-1}),$$

the order term being independent of $\tau$. Thus by theorem 8·1

$$|Dv| \leqslant L' \quad \text{on} \quad \partial\Omega$$

where $L'$ is independent of $\tau$.

It remains to derive an interior estimate for $|Dv|$. We first observe that the coefficients in (61), considered as functions of $x$, are in $C^1(\overline{\Omega})$. Thus, since the boundary data is of class $C^3$, it follows from the Schauder estimates that $v$ is in $C^{2+\lambda}(\overline{\Omega})$ for any $\lambda < 1$. This in turn implies that the coefficients of (61) are in $C^{1+\lambda}(\overline{\Omega})$. Using linear theory once more, we find that $v \in C^{3+\lambda}(\Omega)$.

The hypotheses of theorem 12·3 or of theorem 13·3 are now satisfied by $v$ and by equation (61), with the order terms independent of $\tau$. Hence

$$|Dv| \leqslant M' \quad \text{in} \quad \Omega$$

where $M'$ is independent of $\tau$.

We have thus established that $|v| + |Dv| \leqslant m' + M'$ in $\Omega$; therefore by the fundamental existence theorem, the Dirichlet problem for equation (1) in $\Omega$ is solvable for arbitrary boundary data of class $C^3$.

The differentiability assumptions on the domain, the data, and the functions $\mathscr{A}$ and $\mathscr{B}$ can be removed by a standard approximation and compactness argument. To this end, we observe first that the numbers $m'$ and $M'$ above depend only on $C^2$ norms of the domain and the data, and on the structure stated in the hypothesis of the theorem. This being the case, we can approximate the given problem 'uniformly' by new problems possessing appropriate differentiability conditions; the new problems then have solutions which, by the above argument, satisfy

$$|u|+|Du| \leqslant m'+M'+1, \quad \text{say.}$$

According to the theorem of Ladyzhenskaya & Uraltseva these solutions are then uniformly bounded in $C^{1+\gamma}$. Finally, by linear theory, the solutions are uniformly bounded in $C^{2+\gamma}$ on compact subsets of $\Omega$. Applying Arzela's theorem, we can choose a subsequence of these solutions which converges together with its second derivatives to a solution of the original problem. This completes the proof of theorem 1.

It is not hard to see that the solution which has been constructed is of class $C^1(\overline{\Omega})$, and even (using the theorem of Ladyzhenskaya & Uraltseva) of class $C^{1+\gamma}(\overline{\Omega})$ for an appropriate value of $\gamma$.

The above result includes a main existence theorem of Bernstein (1912, page 464) and Leray (1939, page 281), who required not only $n = 2$ but also various additional restrictions on the behaviour of the functions $\mathscr{A}$ and $\mathscr{B}$ for large values of $p$.

Theorem 2. *Suppose the hypotheses of theorem* 1 *hold, except for the final condition concerning* $\mathscr{B}$ *and* $\mathscr{E}$. *Then the Dirichlet problem for equation* (1) *in* $\Omega$ *is solvable for arbitrary data of class* $C^2$ *provided the diameter of* $\Omega$ *is sufficiently small* (*depending only on the structure of the equation and on a bound for* $\sup|f|$).

*Proof.* As in the proof of theorem 1, we can assume that the domain and the data are of class $C^3$, and that the functions $\mathscr{A}$ and $\mathscr{B}$ are of class $C^2$.

Let $g(u)$ be a non-decreasing $C^2$ function such that

$$g(u) = \begin{cases} -2-c_0 & \text{if} \quad u \leqslant -2-c_0, \\ u & \text{if} \quad -1-c_0 \leqslant u \leqslant 1+c_0, \\ 2+c_0 & \text{if} \quad u \geqslant 2+c_0. \end{cases} \qquad (c_0 = \sup_\Omega |f|)$$

We consider a new equation with coefficient matrix

$$\hat{\mathscr{A}}(x,u,p) = \mathscr{A}(x, g(u), p)$$

and with a similarly defined inhomogeneous term $\hat{\mathscr{B}}$. It will be enough to show that the new equation has a solution $u$ satisfying

$$|u| \leqslant 1+c_0, \quad u = f \quad \text{on} \quad \partial\Omega,$$

for this solution will evidently also solve the Dirichlet problem for the original equation.

Consider a function $v \in C^2(\overline{\Omega})$ such that

$$\hat{\mathscr{A}} D^2 v = \tau \hat{\mathscr{B}} \quad \text{and} \quad v = \tau f \quad \text{on} \quad \partial\Omega.$$

In order to establish the existence of a solution of the new Dirichlet problem it will be enough to show that

$$|v| \leqslant 1+c_0 \quad \text{in} \quad \Omega,$$

for in all other respects the previous proof can be applied unchanged to the present situation.

Since $\mathscr{B} = O(\mathscr{E})$, there exist positive constants $\alpha$ and $C$ such that

$$|\mathscr{B}|/\mathscr{E} \leqslant C \quad \text{for} \quad |p| \geqslant \alpha, \quad |u| \leqslant 2+c_0.$$

Consequently
$$|\hat{\mathscr{B}}|/\hat{\mathscr{E}} \leqslant C \quad \text{for} \quad |p| \geqslant \alpha.$$

Now, by a simple modification of the proof of theorem 6·4, we find that

$$|v| \leqslant c_0 + \frac{1}{C}\log\frac{1}{1-\alpha C\bar{\delta}} \quad \text{in} \quad \Omega$$

provided $\bar{\delta} < 1/\alpha C$, where $\bar{\delta}$ is the diameter of $\Omega$. In fact, if

$$\bar{\delta} \leqslant \frac{1-e^{-C}}{\alpha C}$$

then $|v| \leqslant 1+c_0$, and the proof is complete.

*Remarks*. In both theorems 1 and 2 the conditions on $\mathscr{B}_u$ and $\mathscr{E}_u$ can be weakened

$$\mathscr{B}_u \geqslant -o(\mathscr{E}) \quad \text{and} \quad \mathscr{E}_u = o(\mathscr{E})$$

according to the remark at the end of § 12. Furthermore, it almost goes without saying that the final hypothesis of theorem 1 can be dispensed with entirely in any situation where the solutions of (61) are known to be uniformly bounded in absolute value.

Finally, we note that the size of the domain $\Omega$ required in the conclusion of theorem 2 will be independent of the given boundary conditions whenever the order relation $\mathscr{B} = O(\mathscr{E})$ is uniform over all values of $u$. This holds in particular for the important case when the coefficients $\mathscr{A}$ and $\mathscr{B}$ are independent of $u$.

THEOREM 3. *Let $\Omega$ be a bounded domain in n-dimensional Euclidean space, whose boundary is of class $C^2$. Assume either $n = 2$ or that (60) holds.*

*Suppose there exists a constant $\mu > 0$ such that the invariant function $\mathscr{E}$ satisfies*

$$p\cdot\mathscr{E}_p \leqslant (1-\mu)\,\mathscr{E}, \quad \mathscr{E} \geqslant \mu$$

*at least for all sufficiently large p. Assume further that as p tends to infinity*

$$\mathscr{E}_x, \mathscr{E}_u, \mathscr{E}_p = O(\mathscr{E})$$

*and that the asymptotic formulae (26) and (27) hold with the remainder $\mathscr{R} = \mathscr{C} - \mathscr{C}_0$ satisfying*

$$\mathscr{R}, \mathscr{R}_x, \mathscr{R}_u, |p|\,\mathscr{R}_p = O(\mathscr{E}/|p|).$$

*Finally, suppose that $\mathscr{B}$ satisfies the hypothesis of either theorem 6·2 or 6·3.*

*Then the Dirichlet problem for equation (1) in $\Omega$ is solvable for given $C^2$ boundary values f, provided both conditions*

$$\mathscr{H} > -\mathscr{C}_0(y, f, \nu), \quad \mathscr{I} > \mathscr{C}_0(y, f, -\nu) \tag{62}$$

*are satisfied at each point y of the boundary surface, where $\mathscr{H}$ and $\mathscr{I}$ are the generalized mean curvatures defined in § 9. If $\mathscr{A} - \mathscr{A}_0 = o(\mathscr{E}/|p|)$ the equality sign can be allowed in (62).*

*Proof.* A simplified version of the proof can be given if it is assumed that (62) continues to hold when $\mathscr{C}_0$ and $f$ are replaced by $\tau\mathscr{C}_0$ and $\tau f$, $0 \leqslant \tau \leqslant 1$. Since a large number of special examples have this property (cf. §§ 19–22), it seems worthwhile to carry out the proof first under this additional assumption.

As in the proof of theorem 1, we shall suppose to begin with that the domain and the data are of class $C^3$, and the functions $\mathscr{A}$ and $\mathscr{B}$ are of class $C^2$.

Now let $v \in C^2(\overline{\Omega})$ be a solution of (61) in $\Omega$, such that $v = \tau f$ on $\partial\Omega$. We wish to show that $|v|+|Dv| \leqslant M$ in $\Omega$, where $M$ is independent of $\tau$. By theorem 6·2 or 6·3 we have $|v| \leqslant m'$ in $\Omega$. Moreover, using the special assumption above, the required estimate for $|Dv|$ follows immediately from theorem 9·1 and either theorem 12·2 or 13·2.

Next suppose that $\mathscr{A}-\mathscr{A}_0 = o(\mathscr{E}/|p|)$ and

$$\mathscr{H} \geqslant -\mathscr{C}_0(y,f,\nu), \quad \mathscr{I} \geqslant \mathscr{C}_0(y,f,-\nu). \tag{63}$$

Here we must apply theorem 10·1 rather than theorem 9·1. To this end, it will be sufficient to show that

$$\frac{\|\mathscr{A}-\mathscr{A}_0\|+|\mathscr{C}-\mathscr{C}_0|+|p|^{-1}}{\mathscr{F}} = O(|p|^{-1})$$

as $p \to \infty$, uniformly for $|p_0|$ in any compact set. This in turn follows directly from the hypotheses of the theorem provided

$$\mathscr{F} \geqslant \text{const.}\ \mathscr{E} \quad (\text{const.} > 0)$$

for $|p_0| \leqslant c$ and all sufficiently large values of $p$. (Here $c$ denotes an arbitrary fixed real number, $0 < c < \infty$.)

Now by integrating the condition $p \cdot \mathscr{E}_p \leqslant (1-\mu)\mathscr{E}$ it is clear that

$$\mathscr{E} \leqslant \text{const.}\ |p|^{1-\mu}$$

for all sufficiently large $p$. But then, as in the proof of the lemma in §10,

$$\sigma \mathscr{A}_0 \sigma = 0$$

and
$$\mathscr{F} \geqslant \mathscr{E} + o(\mathscr{E}) \quad \text{for} \quad |p_0| \leqslant c.$$

Consequently $\mathscr{F} \geqslant \frac{1}{2}\mathscr{E}$ for $|p_0| \leqslant c$ and all sufficiently large values of $p$ (it almost goes without saying that we are here considering only a compact set of values of $x$ and $u$, namely $x \in \overline{\Omega}$ and $|u| \leqslant m'$).

In order to remove the differentiability assumptions on the domain, the data, and the functions $\mathscr{A}$ and $\mathscr{B}$, we can apply the approximation and compactness methods used in the proof of theorem 1. In so doing, however, an additional point enters the argument when condition (62) is replaced by the sharper condition (63). Indeed, for the approximating problems in the latter case we can only assert that

$$\mathscr{H} \geqslant -\mathscr{C}_0(y,f,\nu)-\epsilon, \quad \mathscr{I} \geqslant \mathscr{C}_0(y,f,-\nu)-\epsilon \tag{64}$$

though of course $\epsilon$ can be chosen arbitrarily small.

Consequently, theorem 10·1 cannot be applied directly to the approximating problems.† The proof of theorem 10·1, however, continues to hold after a slight modification to take into account the weakened condition (64). To see this, we observe that the principal estimate for $\mathscr{L}(v+b)$ in this proof must be replaced in the present circumstances by

$$\mathscr{L}(v+b) \leqslant h'\mathscr{F}\left\{\frac{h''}{h'^3}+c\left(\frac{1}{h'^2}+\frac{\epsilon}{\mu}+\Phi(h'-c_1)\right)\right\}$$

† A somewhat similar point occurs in the use of theorems 6·2 or 6·3 to obtain a maximum bound for the solutions of the approximating problems. The difficulty here is more apparent than real, however, since one can obviously choose the new problems so that the hypotheses $u\mathscr{B} \geqslant 0$ or $\operatorname{sign} u . \mathscr{C} \geqslant -1/R$ continue to hold for large values of $u$.

(note that $\mathscr{F} \geqslant \frac{1}{2}\mu$ for sufficiently large values of $p$). The remaining part of the proof of theorem 10·1 then carries over essentially unchanged provided that

$$\int_{\alpha}^{\infty} \frac{\mathrm{d}\rho}{1+\rho^2[\epsilon\mu^{-1}+\Phi(\rho-c_1)]} > cM, \tag{65}$$

where (for the purpose at hand)

$$\Phi(\rho) = \text{const.}/\rho, \quad M = c_0+m'+1.$$

Inequality (65) is obvious, however, provided $\epsilon$ is sufficiently small. Thus if the approximating problems are sufficiently near to the given problem the conclusion of theorem 10·1 continues to hold. The argument of theorem 1 can then be carried over word for word to the present case, and theorem 3 is thus proved under the initial assumption on the structure of condition (62).

In order to carry out the proof without particular assumptions concerning condition (62), we must resort to a more refined homotopy than used in (61). To this end, let $k(\tau)$ be a function of class $C^2$ which vanishes for $\tau \leqslant 0$ and equals one for $\tau \geqslant \frac{1}{3}$. Also let

$$k_1 = k_1(\tau) = k(\tau-\tfrac{1}{3}), \quad k_2 = k_2(\tau) = k(\tau-\tfrac{2}{3})$$

and

$$\Theta(p) = k(|p|-1).$$

We now put, for $0 \leqslant \tau \leqslant 1$,

$$\mathscr{B}(x,u,p;\tau) = [\Theta(p)+\{1-\Theta(p)\}k_2]\,\mathscr{B}(x,k_1u+(1-k_1)f,p)$$

and consider the family of equations

$$\mathscr{A}(x,v,Dv)\,D^2v = \mathscr{B}(x,v,Dv;\tau) \tag{66}$$

together with the boundary condition $v = k(\tau)f$. Note in particular that the assumptions stated in the concluding remark of § 1 are fulfilled by this homotopy.

Now let $v \in C^2(\overline{\Omega})$ be a solution of (66) in $\Omega$, such that $v = k(\tau)f$ on $\partial\Omega$. In order to apply the concluding remark of § 1, we wish to show that $|v|+|Dv| \leqslant M$ in $\Omega$, where $M$ is independent of $\tau$. It is easy to see that the hypothesis of theorem 6·2 (or theorem 6·3) continues to hold for equation (66) when $\frac{2}{3} \leqslant \tau \leqslant 1$. Moreover, $\mathscr{B}(x,u,0;\tau) \equiv 0$ for $0 \leqslant \tau \leqslant \frac{2}{3}$. Thus theorem 6·2 remains (is) applicable in this range. Hence a uniform bound for $|v|$ follows at once, say $|v| \leqslant m'$ in $\Omega$, where $m'$ is independent of $\tau$.

It is, furthermore, clear that $\mathscr{C}_0(x,u,\sigma;\tau) = \mathscr{C}_0(x,k_1u+(1-k_1)f,\sigma)$. Hence $\mathscr{H}$, $\mathscr{I}$, and $\mathscr{C}_0(y,f,\pm\nu)$ are invariant under the given homotopy, and conditions (62) continue to hold at each stage. Consequently, under the usual smoothness assumptions on the domain, the data, and the coefficients, theorem 9·1 can be used to bound the gradient of $v$ on $\partial\Omega$ independently of $\tau$. Finally, by theorem 12·1 or 13·1 we obtain the required estimate for $|Dv|$ in $\Omega$. Therefore the existence theorem given in the concluding remark of § 1 can be applied, and the theorem is proved in the case where (62) applies and where the domain, the data, and the functions $\mathscr{A}$ and $\mathscr{B}$ are suitably smooth.

As in the first part of the proof, it remains to consider the case where equality is allowed in (62) and to carry out the usual approximation and compactness arguments. But this can be done exactly as before, and the demonstration of theorem 3 is thus completed.

THEOREM 4. *Suppose the hypotheses of theorem 3 hold, with the exception of the final condition concerning $\mathscr{B}$. Then the conclusion also holds, provided the diameter of $\Omega$ is sufficiently small (depending only on the structure of the equation and a bound for* $\sup|f|$).

*Proof.* The argument of theorem 2 applies until the final paragraph, with obvious modifications for the alternate homotopy used above. For the completion of the proof, we then replace the final part of the argument of theorem 2 with the following reasoning.

Since $\mathscr{C} = \mathscr{C}_0 + \mathscr{R}$ and $\mathscr{E} \leqslant \text{const.}\,|p|^{1-\mu}$ it is evident from the hypotheses of theorem 3 that

$$|\mathscr{C}| \leqslant C \quad \text{for} \quad |p| \geqslant \alpha, \quad |u| \leqslant 2 + c_0$$

where $\alpha$ and $C$ are appropriate positive constants. Consequently, since $\mathscr{E} \geqslant \mu$, we have

$$|\hat{\mathscr{B}}|/\hat{\mathscr{E}} \leqslant C'|p| \quad \text{for} \quad |p| \geqslant \alpha$$

where $C' = C/\mu$. Now, by the proof of theorem 6·4,

$$|v| \leqslant c_0 + \frac{1-\sqrt{(1-2\alpha^2 C'\bar{\delta})}}{\alpha C'} \quad \text{in} \quad \Omega$$

provided $\bar{\delta} \leqslant 1/2\alpha^2 C'$, where $\bar{\delta}$ is the diameter of $\Omega$. In fact, if

$$\bar{\delta} \leqslant \begin{cases} \dfrac{2-\alpha C'}{2\alpha}, & \alpha C' < 1, \\ 1/2\alpha^2 C', & \alpha C' \geqslant 1, \end{cases}$$

then $|v| \leqslant 1 + c_0$, and the proof is complete.

The following result is a variant of theorem 3 in which it is not necessary to introduce the asymptotic formula (26).

THEOREM 5. *Let $\Omega$ be a bounded domain in n-dimensional Euclidean space, whose boundary is of class $C^2$. Assume either $n = 2$ or that (60) holds.*

*Suppose there exists a constant $\mu > 0$ such that*

$$p \cdot \mathscr{E}_p \leqslant (1-\mu)\,\mathscr{E}, \quad \mathscr{E} \geqslant \mu$$

*for all sufficiently large p. Assume further that as p tends to infinity*

$$\mathscr{E}_x,\ \mathscr{E}_u,\ \mathscr{E}_p = O(\mathscr{E})$$

*and that the asymptotic formula (27) holds with the remainder $\mathscr{R} = \mathscr{C} - \mathscr{C}_0$ satisfying*

$$\mathscr{R},\ \mathscr{R}_x,\ \mathscr{R}_u,\ |p|\mathscr{R}_p = O(\mathscr{E}/|p|).$$

*Finally, suppose that $\mathscr{B}$ obeys the hypothesis of either theorem 6·2 or 6·3.*

*Then the Dirichlet problem for equation (1) in $\Omega$ is solvable for given $C^2$ boundary values f, provided that*

$$k > \mp\mathscr{C}_0(y, f, \pm\nu) \tag{67}$$

*at each point y of the boundary, where $k = k(y)$ denotes the minimum normal curvature of the boundary surface at y.*

*If the minimum eigenvalue of $\mathscr{A}$ is bounded below by a positive multiple of $|p|^{-2}$ as p becomes large, then the equality sign can be allowed in (67).*

*Proof.* The first part of the theorem is proved in the same way as the first part of theorem 3, with the exception that theorem 9·2 rather than theorem 9·1 is used to bound $|Dv|$.

Next suppose the minimum eigenvalue of $\mathscr{A}$ is bounded below by a positive multiple of $|p|^{-2}$, and that (67) is replaced by

$$k \geqslant \mp\mathscr{C}_0(y, f, \pm\nu). \tag{68}$$

Here we must apply theorem 10·2 rather than theorem 9·2. To this end, it will be sufficient to show that

$$\frac{|\mathscr{C}-\mathscr{C}_0|+|p|^{-1}}{\mathscr{F}} = O(|p|^{-1})$$

as $p\to\infty$, uniformly for $|p_0|$ in any compact set. This in turn follows directly from the hypotheses of the theorem provided

$$\mathscr{F} \geqslant \text{const.}\,\mathscr{E} \quad (\text{const.} > 0)$$

for $|p_0| \leqslant c$ and all sufficiently large values of $p$.

To prove this, let $\lambda = \lambda(x,u,p)$ denote the minimum eigenvalue of $\mathscr{A}$, and suppose

$$\lambda \geqslant \mu_0/|p|^2 \quad \text{for} \quad |p| \geqslant \alpha,$$

where $\mu_0$ is a positive constant. Then obviously

$$\mathscr{F} \geqslant \tfrac{1}{4}\mu_0$$

for $|p_0| \leqslant c$ and $|p| \geqslant \max(\alpha, 2c)$. It follows that

$$\mathscr{F} \geqslant \frac{\mu_0}{2(\mu_0+8c^2)}\,\mathscr{E}$$

when $\mathscr{E} \leqslant \frac{1}{2}(\mu_0+8c^2)$; on the other hand, when $\mathscr{E} \geqslant \frac{1}{2}(\mu_0+8c^2)$ we have by the lemma of § 8

$$\mathscr{F} \geqslant \tfrac{1}{2}\mathscr{E} - 2c^2 \geqslant \left(\frac{1}{2}-\frac{4c^2}{\mu_0+8c^2}\right)\mathscr{E}$$

and the same inequality holds. This being shown, the second part of the theorem now follows from the same argument used to prove the second part of theorem 3.

*Remark.* The hypothesis requiring $\mathscr{B}$ to satisfy the conditions of either theorem 6·2 or 6·3 can obviously be replaced with various other conditions which provide *a priori* bounds for the magnitude of a solution. With this remark taken into account, theorem 5 includes as a corollary the result stated on page 282 of Leray's memoir on the Dirichlet problem.

A final example of interest concerns the equation

$$(1+u_y^2)\,u_{xx} - 2u_x u_y u_{xy} + (1+u_x^2)\,u_{yy} = cu(1+u_x^2+u_y^2)^{\frac{3}{2}}$$

($c = \text{const.} > 0$) which governs the free surface of a stationary fluid under the combined action of gravity and surface tension. Since

$$\mathscr{C} = \frac{cu(1+|p|^2)^{\frac{3}{2}}}{|p|(2+|p|^2)}, \quad \mathscr{E} = \frac{|p|^2}{2+|p|^2}, \quad \lambda = \frac{1}{2+|p|^2}$$

and

$$\mathscr{C}_0 = cu, \quad \mathscr{R} = -\tfrac{1}{2}cu(|p|^{-2}+\ldots),$$

it is easily checked that the hypotheses of theorem 5 are satisfied (theorem 6·2 at the final step). Thus the Dirichlet problem is solvable for given $C^2$ boundary values $f$ provided

$$\kappa \geqslant c\,|f|$$

at each point of the boundary, where $\kappa$ denotes the ordinary curvature of the bounding curve.

While various further results are suggested by the foregoing summary, these may be left to the reader for formulation. In chapter IV we shall consider a number of special equations whose discussion requires more detailed individual analysis.

## Chapter III

In this chapter we shall treat various cases of the non-solvability of Dirichlet's problem for smooth data. It will be shown that the results of chapter II are in many ways best possible, in that, when their principal hypotheses fail to hold, one may construct boundary data for which the Dirichlet problem cannot be solved.

We shall assume throughout that (1) is normalized by the condition trace $\mathscr{A} = 1$. Also, the invariants $\mathscr{C}$ and $\mathscr{E}$ will retain the meanings assigned in § 5.

### 15. A variation of the maximum principle

The results of the chapter depend on an interesting form of the maximum principle, in which the hypothesis $u \leqslant \omega$ is required only on a subset of the boundary. For the purposes of stating this result, as well as for later use, we introduce some simple terminology. Let $\Gamma$ be a domain in $n$-dimensional space with boundary $\partial\Gamma$. If $P$ is a point of $\partial\Gamma$ and $N$ is an $n$-dimensional neighbourhood of $P$, the set $N' = \partial\Gamma \cap N$ is called a *boundary neighbourhood* of $P$. A subset of $\Gamma$ is called *open* if it contains a boundary neighbourhood of each of its points.

Theorem 1. *Let $\Gamma$ be a bounded domain in n-dimensional Euclidean space, whose boundary contains an open subset $\gamma$ of class $C^1$. Let $\omega(x)$ be a function in $C(\overline{\Gamma}) \cap C^2(\Gamma)$ satisfying*

$$\mathscr{L}(\omega+b) \leqslant 0 \quad \textit{in} \quad \Gamma$$

*for all positive constants $b$, and such that $\partial\omega/\partial n = -\infty$ at each point of $\gamma$ (here $\vec{n}$ denotes the normal direction into $\Gamma$.)*

*Suppose that $u \in C(\overline{\Gamma}) \cap C^2(\Gamma+\gamma)$ is a solution of (1) in $\Gamma$ such that $u \leqslant \omega$ on $\partial\Gamma-\gamma$. Then $u \leqslant \omega$ in $\Gamma$.*

The underlying idea of theorem 1 was known and used by Bernstein (1910, page 243; 1912, pages 465–469). More recently, Leray (1939) and Finn (1965) have applied the result of theorem 1 in essentially the form in which it is stated here. For completeness, we give the simple proof.

Suppose for contradiction that the conclusion is not true. Then by theorem 6·1 and the fact that $u \leqslant \omega$ on $\partial\Gamma-\gamma$, the maximum of the function $u-\omega$ must occur at some point $P$ on $\gamma$. This is impossible, however, since at $P$ we have

$$\frac{\partial}{\partial n}(u-\omega) = \frac{\partial u}{\partial n} - \frac{\partial \omega}{\partial n} = \infty$$

so that $u-\omega$ takes values in $\Gamma$ which are larger than its value at $P$.

### 16. Irregularly elliptic equations

For the purposes of classifying equations for which the Dirichlet problem is not always solvable, it is convenient to introduce some further terminology. We say that equation (1) is *irregularly elliptic* provided that

$$|\mathscr{C}| \to \infty \quad \text{as} \quad p \to \infty, \quad \text{uniformly in } u \tag{69}$$

and $$|\mathscr{C}|/\mathscr{E} \geqslant \Psi(|p|) \quad \text{for} \quad |u| \geqslant M, \quad |p| \geqslant l, \tag{70}$$

where $M$ and $l$ are positive constants and $\Psi(\rho)$ is a positive continuous function satisfying the condition

$$\int^{\infty} \frac{\mathrm{d}\rho}{\rho^2 \Psi(\rho)} < \infty. \tag{71}$$

Evidently the classes of regularly elliptic and irregularly elliptic equations are disjoint, though obviously there are equations which belong to neither class.

To illustrate the concept of irregular ellipticity, suppose that $\mathscr{E} \geqslant \mu|p|$ when $|p| \geqslant 1$. Then (1) is irregularly elliptic provided that

$$|\mathscr{B}|/\mathscr{E} \geqslant (\log|p|)^{1+\theta} \quad \text{for large } u \text{ and } p, \tag{72}$$

where $\theta$ is a positive constant. In particular, a uniformly elliptic equation is irregularly elliptic whenever

$$|\mathscr{B}| \geqslant |p|^2 (\log|p|)^{1+\theta} \quad \text{for} \quad \text{large } u \text{ and } p.$$

Condition (72) should be compared with the corresponding relation (22), which guarantees regular ellipticity.

The following theorem relates the concept of irregular ellipticity with the non-solvability of Dirichlet's problem.

Theorem 1. *Let $\Omega$ be a bounded domain in n-dimensional Euclidean space, and suppose that equation* (1) *is irregularly elliptic. Then there exists $C^{\infty}$ boundary data such that the Dirichlet problem for equation* (1) *in $\Omega$ has no solution.*

As a consequence of theorem 1, we see that for equations having a well-defined genre the Dirichlet problem is generally not well set when the degree of the inhomogeneous term $\mathscr{B}$ is greater than $\max(1, 2-g)$. Conversely, when the degree is at most this large the results of § 14 become applicable.

*Proof of theorem* 1. We may obviously assume without loss of generality that (69) and (70) hold with the absolute value signs removed from $\mathscr{C}$.

Now let $P$ be any fixed boundary point of $\Omega$ at which there is an internally touching sphere $\Sigma$. We shall show that there exists a boundary neighbourhood $N'$ of $P$ and a constant $c$ such that if $u$ is a solution of the Dirichlet problem for equation (1) in $\Omega$, then

$$u \leqslant \max(m, M) + c \quad \text{at } P \tag{73}$$

where $m$ denotes the supremum of $u$ on $\partial\Omega - N'$ and $M$ is defined by (70).

Assuming this result for the moment, in order to prove theorem 1 it then suffices to choose smooth boundary values $f$ such that $f \equiv 0$ on $\partial\Omega - N'$ and $f > c + M$ at $P$.

Turning to the proof of (75), let $2a$ be the radius of $\Sigma$, let $\bar{\delta}$ be the diameter of $\Omega$, and let $r$ denote distance from $P$. Consider the function

$$\omega(x) = M + h(r), \quad a \leqslant r \leqslant \bar{\delta},$$

where $h$ is twice continuously differentiable for $a < r \leqslant \bar{\delta}$, and

$$h'(a) = -\infty, \quad h(\bar{\delta}) = 0, \quad h'(r) \leqslant -l. \tag{74}$$

An easy calculation yields

$$\mathscr{L}(\omega + b) \leqslant \mathscr{E}(x, \omega + b, D\omega)\,(h''/h'^2) - \mathscr{B}(x, \omega + b, D\omega)$$

where $b$ is an arbitrary positive constant. Now using (70) we obtain

$$\mathscr{L}(\omega+b) \leqslant \mathscr{E}h'\{h''/h'^3+\Psi(|h'|)\}$$

(note here that $\omega+b \geqslant M$ and $|D\omega| = |h'| \geqslant l$).

As a preparatory step for the application of theorem 15·1, we shall choose $h$ so that the preceding expression is non-positive. To this end, put

$$C = \max\left(1, (\bar{\delta}-a)\Big/\int_l^\infty \frac{d\rho}{\rho^3\Psi(\rho)}\right)^\dagger$$

and let $\alpha$ be a constant defined by

$$\begin{cases} \displaystyle\int_\alpha^\infty \frac{d\rho}{\rho^3\Psi(\rho)} = \bar{\delta}-a & \text{if} \quad C = 1, \\ \alpha = l & \text{if} \quad C = (\bar{\delta}-a)\Big/\displaystyle\int_l^\infty \frac{d\rho}{\rho^3\Psi(\rho)}. \end{cases}$$

Clearly $\alpha \geqslant l$. Now let $h$ and $r$ be parametrically related by

$$h = C\int_\alpha^\rho \frac{d\rho}{\rho^2\Psi(\rho)}, \quad r = a + C\int_\rho^\infty \frac{d\rho}{\rho^3\Psi(\rho)}$$

where $\alpha \leqslant \rho \leqslant \infty$. One easily checks that $h' = -\rho \leqslant -l$ and that (74) holds. Moreover

$$\frac{h''}{h'^3}+\Psi(|h'|) = \frac{C-1}{C}\Psi(|h'|) \geqslant 0$$

so that $\mathscr{L}(\omega+b) \leqslant 0$.

We may now apply theorem 15·1 to the domain $\Gamma = \Omega \cap \{r > a\}$ and the open boundary set $\gamma = \Omega \cap \{r = a\}$. Putting

$$N' = \partial\Omega \cap \{r < a\}$$

then yields at once

$$u \leqslant \max(m, M) + C\int_l^\infty \frac{d\rho}{\rho^2\Psi(\rho)} \equiv m^* \tag{75}$$

in $\Gamma$, where $m$ is the supremum of $u$ on $\partial\Omega - N'$.

Although (75) supplies a bound for $u$ on the set $\Omega \cap \{r = a\}$, a further argument is needed to obtain an estimate for $u$ at $P$. Let $\epsilon$ be an arbitrary real number between 0 and $a$, and let $s$ denote distance from the *centre* of $\Sigma$. Consider the function

$$\omega(x) = m^* + h(s), \quad a \leqslant s \leqslant 2a-\epsilon,$$

where $h$ is twice continuously differentiable for $a \leqslant s < 2a-\epsilon$, and

$$h'(2a-\epsilon) = \infty, \quad h(a) = 0, \quad h'(s) \geqslant l^*.$$

(No confusion should result from the fact that $h(r)$ in the earlier part of the proof and $h(s)$ in the present part represent different functions.) An easy calculation now gives

$$\mathscr{L}(\omega+b) \leqslant \mathscr{E}(x, \omega+b, D\omega)\frac{h''}{h'^2}+\frac{h'}{a}-\mathscr{B}(x, \omega+b, D\omega)$$

$$= \mathscr{E}h'\left\{\frac{h''}{h'^3}+\frac{a^{-1}-\mathscr{C}}{\mathscr{E}}\right\}.$$

† Note that the integral converges in virtue of condition (71).

Since $\mathscr{C}\to\infty$ as $p\to\infty$ there exists a positive constant $l_1 = l_1(a)$ such that

$$\mathscr{C} \geqslant 2a^{-1} \quad \text{for} \quad |p| \geqslant l_1.$$

With the help of (70), it now follows that

$$\mathscr{L}(\omega+b) \leqslant \mathscr{E}h'\left\{\frac{h''}{h'^3}-\tfrac{1}{2}\Psi(h')\right\}$$

provided $l^* = \max(l, l_1)$.

We now choose $h$ so that the expression in braces in non-positive. The details are essentially the same as in the first part of the proof (replace $l$ by $l^*$, $\bar{\delta}-a$ by $a-\epsilon$, and $r-a$ by $2a-\epsilon-s$) and consequently may be omitted.

The proof is completed by applying theorem 15·1 to the domain

$$\Gamma^* = \{r<a\} \cap \{s<2a-\epsilon\}$$

and the open boundary set $\gamma^* = \{r<a\}\cap\{s = 2a-\epsilon\}$. Indeed, according to (75), we have $u \leqslant m^*$ on the remaining boundary set $\{r = a\}\cap\{s\leqslant 2a-\epsilon\}$. Hence by theorem 15·1 there results

$$u \leqslant m^* + C^*\int_{l^*}^{\infty}\frac{\mathrm{d}\rho}{\rho^2\Psi(\rho)}, \quad C^* = \max\left(2, a\bigg/\int_{l^*}^{\infty}\frac{\mathrm{d}\rho}{\rho^3\Psi(\rho)}\right)$$

in $\Gamma^*$. Letting $\epsilon\to 0$ and using the continuity of $u$ yields the same inequality at $P$. This establishes (73), and thus completes the demonstration of the theorem.

Bernstein (1912, pages 465–469) proved for the case of two independent variables that if equation (1) has a well defined integer valued genre $g\leqslant 2$, and if, moreover:

(i) $\mathscr{A}$ and $\mathscr{B}$ are analytic and have certain special expansion properties for large values of $p$,

(ii) $\mathscr{A} = \mathscr{A}(x,p)$ and $\partial\mathscr{B}/\partial u \geqslant 0$,

(iii) $\mathscr{B}/\mathscr{E} \geqslant \text{const.}\,|p|$ for large $p$ (const. $>0$),

then there exist convex domains $\Omega$ and smooth boundary data $f$ for which the Dirichlet problem is *not* solvable. The cases $g = 0$, 1 of Bernstein's theorem are easily seen to follow from our results, for in fact theorem 1 shows that under the hypothesis

(iii′) $\mathscr{B}/\mathscr{E} \geqslant \text{const.}\,|p|^{\theta}$, $\mathscr{E} \geqslant \text{const.}\,|p|$

alone, the Dirichlet problem is not generally solvable for *any* domain, no matter how it is chosen. The case $g = 2$ of Bernstein's theorem falls more properly under the work of succeeding sections, and is in fact a special case of the following theorem 18·3.

## 17. Singularly elliptic equations

In the preceding section, attention was given to the effect of a large inhomogeneous term on the solvability of Dirichlet's problem. Here we shall consider a second type of non-regularly elliptic equation, in which the regular ellipticity now fails due to the structure of the coefficient matrix $\mathscr{A}$ rather than due to the forcing effect of the inhomogeneous term $\mathscr{B}$.

We say that equation (1) is *singularly elliptic* provided that

$$1/\mathscr{E} \geqslant \Psi(|p|) \quad \text{for} \quad |p| \geqslant 1 \tag{76}$$


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where $\Psi(\rho)$ is a positive continuous function satisfying the condition

$$\int^{\infty} \frac{d\rho}{\rho^2 \Psi(\rho)} < \infty.$$

In parallel with the comment in the previous section, it is evident that the class of singularly elliptic equations is disjoint from the class of regularly elliptic equations, though again there are equations which belong to neither class. If equation (1) has a well-defined genre $g$, then clearly a necessary and sufficient for singular ellipticity is that $g > 1$.

In this section, as in the earlier §§ 9 and 10, it is the curvatures of the boundary surface which play a crucial role, and it is the failure of the recurrent condition (29), (35), (62) which leads to non-existence. It will be convenient to defer the main results until a series of lemmas have been proved. From § 12 we recall the definition $\mathscr{E}^* = \mathscr{E}/|p|^2$.

Lemma 1. *Let $u$ be a solution of the Dirichlet problem for equation* (1) *in a bounded domain $\Omega$. Suppose that* (1) *is singularly elliptic. Assume also that*

$$\mathscr{E}^*(x, u, p) \leqslant 1-\mu$$

*for some positive constant $\mu$, and that*

$$\mathscr{B}(x, u, p) \geqslant 0 \quad \textit{for} \quad u \geqslant M.$$

*Let $P$ be a point of $\partial\Omega$ and let $m$ be the supremum of $u$ on the boundary set $\partial\Omega \cap \{r \geqslant a\}$, where $r$ denotes distance from $P$.*

*Then for any given $\epsilon > 0$ we have*

$$u \leqslant \max(m, M) + \epsilon \quad \textit{on} \quad \Omega \cap \{r = a\}$$

*for all sufficiently small values of $a$ (depending only on $\epsilon$, the diameter of $\Omega$, and the structure of the equation).*

*Proof.* As in the previous section, we consider a comparison function of the form

$$\omega(x) = M + h(r), \quad a \leqslant r \leqslant \bar{\delta},$$

where $h$ is twice continuously differentiable for $a < r \leqslant \bar{\delta}$, and

$$h'(a) = -\infty, \quad h(\bar{\delta}) = 0, \quad h'(r) \leqslant 0.$$

An easy calculation gives $\quad \mathscr{L}(\omega + b) = \mathscr{E}\dfrac{h''}{h'^2} + (1-\mathscr{E}^*)\dfrac{h'}{r} - \mathscr{B},$

where $b$ is an arbitrary positive constant and the arguments of $\mathscr{B}$, $\mathscr{E}$ and $\mathscr{E}^*$ are $x$, $\omega + b$, and $D\omega$. Hence, using the conditions $\mathscr{B} \geqslant 0$ and $\mathscr{E}^* \leqslant 1-\mu$, we obtain

$$\mathscr{L}(\omega + b) \leqslant \mathscr{E}\frac{h''}{h'^2} + \mu\frac{h'}{r}.$$

Now define $\Xi(\rho) \equiv \rho^{-2}$ if $0 < \rho \leqslant 1$ and $\Xi(\rho) \equiv \Psi(\rho)$ if $\rho > 1$, where $\Psi(\rho)$ is the given upper bound for $1/\mathscr{E}$ in the definition of singular ellipticity. Clearly

$$\int_0^{\infty} \frac{d\rho}{\rho^2 \Xi(\rho)} < \infty.$$

For $0 < \alpha < \infty$, put
$$\chi(\alpha) = \int_\alpha^\infty \frac{d\rho}{\rho^3 \Xi(\rho)}.$$

Evidently $\chi(\alpha)$ is monotonically decreasing and has range $(0,\infty)$. Thus there exists a well-defined inverse function $\eta(\beta)$, also monotonically decreasing and with range $(0,\infty)$. We have, furthermore,
$$\int_0^\infty \eta(\beta)\, d\beta = \int_0^\infty \chi(\alpha)\, d\alpha < \infty,$$
as is easily seen by an application of Fubini's theorem.

We may now define
$$h(r) = \int_r^{\bar\delta} \eta\left(\mu \log \frac{t}{a}\right) dt.$$

One checks easily that
$$\frac{h''}{h'^3} = -\frac{\mu \Xi(-h')}{r}, \quad |D\omega| = -h';$$
thus $\mathscr{L}(\omega + b) \leqslant 0$ since $\mathscr{E} \leqslant 1/\Xi$.

This much being shown, it follows by applying theorem 15·1 to the domain $\Omega \cap \{r > a\}$ that
$$u \leqslant \max(m, M) + h(a) \quad \text{in} \quad \Omega \cap \{r > a\}.$$

Since
$$h(a) = \int_a^{\bar\delta} \eta\left(\mu \log \frac{t}{a}\right) dt = a \int_1^{\bar\delta/a} \eta(\mu \log t)\, dt \quad (<\infty),$$
we find by L'Hôpital's rule
$$\lim_{a\to 0} h(a) = \lim_{a \to 0} \bar\delta \eta(\mu \log(\bar\delta/a)) = 0$$
and the lemma is proved.

This proof is closely related to one given by Nitsche (1965) in his discussion of the maximum principle for equations of positive genre.

LEMMA 2. *Let* (1) *be singularly elliptic, and suppose that the asymptotic formula* (26) *holds. Then*
$$\sigma \mathscr{A}_0 \sigma = 0.$$
*Moreover, corresponding to any compact set of values $x$, $u$ there exists a positive constant $\mu$ such that*
$$\mathscr{E}^* \leqslant 1 - \mu.$$

*Proof.* We first show that $\sigma \mathscr{A}_0 \sigma = 0$. Indeed, since $\mathscr{E} = p\mathscr{A}p$ we find from (26) and (76) that
$$\sigma \mathscr{A}_0 \sigma = \frac{\mathscr{E}}{|p|^2} + o(1) \leqslant \frac{1}{|p|^2 \Psi} + o(1).$$

Since the first term on the right-hand side is integrable, there must be a sequence of values of $|p|$ for which it tends to zero. Keeping $\sigma$ fixed and letting $p$ tend to infinity on this sequence then proves the first part of the lemma.

This being shown, we now observe that $\mathscr{E}^* = \sigma \mathscr{A}_0 \sigma + o(1)$, whence $\mathscr{E}^* \to 0$ as $p \to \infty$. Since $\mathscr{E}^* < 1$ on any compact set of values of $x$, $u$, $p$, the proof is completed.

LEMMA 3. *Let $u$ be a solution of the Dirichlet problem for equation* (1) *in a domain $\Omega$ whose boundary is of class $C^2$.*

*Suppose that* (1) *is singularly elliptic, and that the asymptotic formulae* (26) *and* (27) *hold with the error terms bounded independently of u. Let P be a point of* $\partial\Omega$ *where*

$$\mathscr{I} < \mathscr{C}_0(y, f, -\nu) \tag{77}$$

*and let m′ be the supremum of u on the set* $\Omega \cap \{r = a\}$.

*Then there exists a constant* $\epsilon_0$, *depending only on* $f(P)$, *on the structure of the equation, and on a lower bound for the difference in* (77), *such that for fixed* $\epsilon' \leqslant \epsilon_0$ *the condition*

$$f(P) - 5\epsilon' \leqslant m' \leqslant f(P) - \epsilon'$$

*cannot occur when* $a \leqslant a_0$; *here* $a_0$ *depends on the same quantities as* $\epsilon_0$ *and also on the domain and on* $\epsilon'$.

*Proof.* To distinguish the values of $\mathscr{I}$, $f$, and $\nu$ at $P = (\bar{y})$ we shall henceforth write $\mathscr{I}(\bar{y})$, $\bar{f}$, and $\bar{\nu}$. Then in view of (77) there exists a positive constant $\theta$ such that

$$\mathscr{I}(\bar{y}) \leqslant \mathscr{C}_0(\bar{y}, \bar{f}, -\bar{\nu}) - 5\theta. \tag{78}$$

It is geometrically evident (and easy to prove using a modified Dupin indicatrix) that there exists a quadric surface $S$ tangent to $\partial\Omega$ at $P$ such that

(i) $S$ has a unique parallel projection onto the tangent plane at $P$,

(ii) the point set $S \cap \{r \leqslant a\}$ is contained in the closure of $\Omega$ for all sufficiently small values of $a$, and

(iii) the generalized mean curvature $\mathscr{I}_S$ of the surface $S$ satisfies the condition

$$\mathscr{I}_S(\bar{y}) \leqslant \mathscr{I}(\bar{y}) + \theta. \tag{79}$$

Our purpose is to apply theorem 15·1 somewhat as in the second part of the proof of theorem 16·1, but with the surface $S$ playing the part of the internal tangent sphere $\Sigma$. To this end, let $S^+$ be the half space of points $x$ on the same side of $S$ as the normal vector $\nu$ at $P$. For points $x$ in $S^+$ we define $d = d(x)$ to be the distance from $x$ to $S$.

Now consider the point set

$$\Gamma = \{r < a\} \cap \{d > \epsilon\}, \quad \epsilon \text{ small.}$$

Clearly, if $a$ is suitably small (say $a \leqslant a_1$) then the set $\Gamma$ is contained in $\Omega$. We may assume, moreover, that $d(x)$ is of class $C^2$ in $\Gamma$, if necessary by making $a_1$ somewhat smaller. For convenience the geometric situation is shown in figure 1, the domain $\Gamma$ being indicated by diagonal shading.

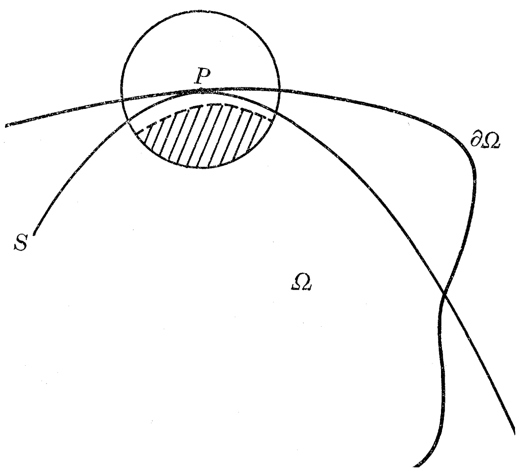


FIGURE 1

For $x$ contained in $\bar{\Gamma}$, and $a < a_1$, we define

$$\omega(x) = m' + h(d), \quad \epsilon \leqslant d \leqslant a,$$

where $h$ is twice continuously differentiable for $\epsilon < d \leqslant a$ and

$$h'(\epsilon) = -\infty, \quad h(a) > 0, \quad h'(d) \leqslant -\alpha, \tag{80}$$

$\alpha$ being a positive constant whose value will be specified later. From lemma 4·1 one finds easily that

$$\mathscr{L}(\omega+b) = \mathscr{E}h''/h'^2 - \mathscr{J}h' - \mathscr{B}, \quad x \in \Gamma, \tag{81}$$

where the arguments of $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{E}$ are $x$, $\omega+b$, and $p = \nu h'$. (It almost goes without saying that the normal vector $\nu$, and the principal directions and principal curvatures which appear in the formula for $\mathscr{J}$, are calculated at the point $y = y(x)$ on $S$ nearest to $x$.)

As in the proofs of chapter II, it will be necessary to estimate the various terms in the preceding inequality. To begin with, using the definition of $\mathscr{C}$ and the asymptotic formula (27), we obtain

$$\begin{aligned}-\mathscr{B}(x, \omega+b, p) &= h'\mathscr{C}(x, \omega+b, p) \\ &= h'\{\mathscr{C}_0(x, \omega+b, -\nu) - o(1)\} \leqslant h'\{\mathscr{C}_0(x, m', -\nu) - o(1)\},\end{aligned}$$

since $\mathscr{C}_0$ is increasing in its second argument and $h'$ is negative (the expression $o(1)$ tends uniformly to zero as $\alpha$ tends to infinity).

We shall assume from here on that $\bar{f} = f(P) \leqslant m' + 5\epsilon'$. Thus to complete the proof it must be shown that $m' > f(P) - \epsilon'$ provided that $\epsilon' \leqslant \epsilon_0$ and $a$ is sufficiently small.

Under the condition $\bar{f} \leqslant m' + 5\epsilon'$, we assert that

$$\mathscr{C}_0(x, m', -\nu) \geqslant \mathscr{C}_0(x, \bar{f}, -\nu) - O(\epsilon').$$

This is obvious if $\bar{f} \leqslant m'$, since $\mathscr{C}_0$ is increasing. On the other hand, when $m' \leqslant \bar{f} \leqslant m' + 5\epsilon'$ the argument $m'$ differs from $\bar{f}$ at most by $5\epsilon'$, and the same result holds (it will be assumed that $\epsilon_0 \leqslant 1$). Since, moreover,

$$\mathscr{C}_0(x, \bar{f}, -\nu) \geqslant \mathscr{C}_0(\bar{y}, \bar{f}, -\bar{\nu}) - O(a)$$

we have finally

$$-\mathscr{B}(x, \omega+b, p) \leqslant h'\{\mathscr{C}_0(\bar{y}, \bar{f}, -\bar{\nu}) - o(1) - O(a) - O(\epsilon')\}.$$

Similarly, from the asymptotic formula (26),

$$\mathscr{J} = \sum_1^{n-1} \frac{\lambda_i \mathscr{A} \lambda_i}{1 - k_i d} k_i = \mathscr{J}_S(\bar{y}) + o(1) + O(a), \tag{82}$$

where we have also made use of the fact that $\nu A \nu = 0$ (see lemma 2).

Substituting the preceding two estimates into (81), and using (78) and (79), now yields

$$\mathscr{L}(\omega+b) \leqslant \mathscr{E}h''/h'^2 + h'\{4\theta - o(1) - O(a) - O(\epsilon')\}.$$

We choose $\epsilon_0$ so that the term $O(\epsilon')$ is less than $\theta$ when $\epsilon' \leqslant \epsilon_0$. Then for $\alpha$ sufficiently large and $a$ sufficiently small, say $\alpha \geqslant \alpha_0$ and $a \leqslant a_2$, we have

$$\mathscr{L}(\omega+b) \leqslant \mathscr{E}(h''/h'^2) + \theta h'.$$



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Now let $a_3$ be defined by the condition

$$a_3 = \frac{1}{\theta}\int_{\alpha_0}^{\infty}\frac{\mathrm{d}\rho}{\rho^3\Psi(\rho)}$$

(we may of course assume $\alpha_0 \geqslant 1$ so that the integral converges). Then for values of $a$ satisfying the condition

$$a \leqslant \min(a_1, a_2, a_3),$$

we define $h(d)$ parametrically by the formulae

$$h = \frac{1}{\theta}\int_{\alpha}^{\rho}\frac{\mathrm{d}\rho}{\rho^2\Psi(\rho)}, \quad d = \epsilon + \frac{1}{\theta}\int_{\rho}^{\infty}\frac{\mathrm{d}\rho}{\rho^3\Psi(\rho)},$$

where $\alpha \leqslant \rho \leqslant \infty$ and where $\alpha$ satisfies the relation

$$a = \frac{1}{\theta}\int_{\alpha}^{\infty}\frac{\mathrm{d}\rho}{\rho^3\Psi(\rho)}. \tag{83}$$

Evidently $\alpha \geqslant \alpha_0$ and conditions (80) hold. Moreover, as one easily checks, $\mathscr{L}(\omega + b) \leqslant 0$ in $\Gamma$.

This being shown, a direct application of theorem 15·1 now implies

$$u \leqslant m' + \frac{1}{\theta}\int_{\alpha}^{\infty}\frac{\mathrm{d}\rho}{\rho^2\Psi(\rho)}$$

in $\Gamma$. Letting $\epsilon \to 0$ yields the same inequality in the limit set $\{r < a\} \cap \{d > 0\}$. Consequently, since $P$ is a boundary point of this set, it follows that

$$f(P) \leqslant m' + \frac{1}{\theta}\int_{\alpha}^{\infty}\frac{\mathrm{d}\rho}{\rho^2\Psi(\rho)}$$

with $\alpha$ defined by (83) and $a \leqslant \min(a_1, a_2, a_3)$.

Clearly $\alpha$ tends to infinity as $a$ tends to zero. Therefore, by choosing $a$ even smaller if necessary, we can make $f(P) < m' + \epsilon'$, completing the proof.

*Remarks.* 1. A similar result can be proved for the generalized mean curvature $\mathscr{H}$. In this case the hypothesis (77) must be replaced by

$$\mathscr{H} < -\mathscr{C}_0(y, f, \nu)$$

and the final condition should correspondingly be changed to

$$f(P) + \epsilon' \leqslant m'' \leqslant f(P) + 5\epsilon'$$

where $m''$ denotes the infimum of $u$ on the set $\Omega \cap \{r = a\}$.

2. If $\mathscr{C}_0$ is independent of $u$, then there is clearly no need to introduce the restriction $\bar{f} \leqslant m' + 5\epsilon'$ in the proof. The conclusion of lemma 3 can then be simplified to the following statement:

*For any given $\epsilon > 0$ we have*

$$f(P) \leqslant m' + \epsilon$$

*for all sufficiently small values of $a$ (depending only on $\epsilon$, $\Omega$, the structure of the equation, and a lower bound for the difference in (77)).*

## 18. Non-existence theorems for singularly elliptic equations

The lemmas of the preceding section can be used to derive several non-existence theorems for singularly elliptic equations. The simplest case occurs when $\mathscr{C}_0$ is independent of $u$. We phrase the result for the invariant $\mathscr{I}$, though obviously a similar result holds for $\mathscr{H}$.

Theorem 1. *Suppose equation* (1) *is singularly elliptic and that the asymptotic formulae* (26) *and* (27) *hold. Assume, furthermore, that* $\mathscr{C}_0$ *is independent of* $u$ *and that* $\mathscr{B} \geqslant 0$ *for* $u \geqslant 0$.

*Let* $\Omega$ *be a bounded domain whose boundary is of class* $C^2$. *If the* (*geometric*) *condition*

$$\mathscr{I} \geqslant \mathscr{C}_0(y, -\nu) \tag{84}$$

*fails at a single point* $y$ *of the boundary surface, then there exists smooth boundary data having arbitrarily small absolute value for which no solution of the Dirichlet problem is possible.*

*Proof.* Let $P$ be a point where (84) fails, and let $\epsilon$ and $a$ be fixed positive constants (with $\epsilon \leqslant 1$). Consider smooth boundary values $f$ such that $f \equiv 0$ on $\partial\Omega \cap \{r \geqslant a\}$ while $f = 3\epsilon$ at $P$. We shall show that if $a$ is chosen sufficiently small (depending on $\epsilon$) then the corresponding Dirichlet problem has no solution.

Suppose in fact that a solution $u$ did exist for the given data $f$. Since $\mathscr{B} \geqslant 0$ for non-negative values of its second argument, it follows from theorem 6·2 that

$$u \leqslant 3\epsilon \quad \text{in} \quad \Omega.$$

We now wish to apply lemma 1. Since the hypothesis $\mathscr{E}^* \leqslant 1-\mu$ may not hold, it is necessary to add some comments. In particular, we observe that the comparison function $\omega + b$ in the proof of lemma 1 need take values only in the closed interval $[0, 3]$. According to lemma 2, however, there exists a constant $\mu$ such that $\mathscr{E}^* \geqslant 1-\mu$ when $x \in \overline{\Omega}$ and $u \in [0, 3]$. The *proof* of lemma 1 therefore carries over unchanged to the present situation. This being the case, it follows that if $a$ is sufficiently small, say $a \leqslant a(\epsilon)$, then

$$u \leqslant \epsilon \quad \text{on} \quad \Omega \cap \{r = a\}.$$

By the same argument, we are entitled to apply lemma 3 in the present circumstances, even though (26) and (27) are not assumed to hold with the error terms bounded independently of $u$. Accordingly, taking into account the remark at the end of the previous section, we have

$$u(P) \leqslant m' + \epsilon \leqslant 2\epsilon$$

when $a$ is suitably small, say $a \leqslant a_0(\epsilon)$. In summary, if $a \leqslant \min\,(a(\epsilon), a_0(\epsilon))$ then the Dirichlet problem can have no solution for the given boundary values $f$. Since $\epsilon$ can be chosen arbitrarily small, this completes the proof. (For the special case of the minimal surface equation theorem 1 is due to Jenkins & Serrin (1968); cf. also Bernstein (1912) and Finn (1965). The above proof is a generalization of the work of these authors.)

The result of theorem 1 strikingly illustrates the primary importance of the geometry of the boundary surface in the study of Dirichlet's problem for singularly elliptic equations.

When $\mathscr{C}_0$ depends on $u$, the results are not quite as sharp, since the boundary data itself enters into the solvability conditions $\mathscr{H} \geqslant -\mathscr{C}_0(y, f, \nu)$ and $\mathscr{I} \geqslant \mathscr{C}_0(y, f, -\nu)$. Nevertheless, these conditions are essentially best possible, in the sense that *if either one of them fails globally for given boundary values* $f$, *then the corresponding Dirichlet problem is badly posed.* The precise result is as follows.

THEOREM 2. *Suppose that equation* (1) *is singularly elliptic and that the asymptotic formulae* (26) *and* (27) *hold. Assume furthermore that* $\mathscr{B} \geqslant 0$ *for* $u \geqslant 0$.

*Let* $\Omega$ *be a bounded domain whose boundary is of class* $C^2$, *and consider arbitrary continuous boundary data* $f$ *such that*

$$\mathscr{I} < \mathscr{C}_0(y, f, -\nu), \quad f \geqslant 0 \tag{85}$$

*at each point* $y$ *of the boundary surface. Then there exists smooth boundary data, whose difference from* $f$ *has arbitrarily small absolute value, for which no solution of the Dirichlet problem is possible.*

*Proof.* Let $P$ be a point of $\partial\Omega$ where $f$ takes its greatest value, and let $\epsilon$ and $a$ be fixed positive numbers with $\epsilon \leqslant 1$. Consider smooth boundary data $g$ such that

$$|g - f| \leqslant \epsilon \quad \text{on} \quad \partial\Omega \cap \{r \geqslant a\} \tag{86}$$

and
$$g = f + 3\epsilon \quad \text{at} \quad P.$$

We shall show that if $\epsilon$ is suitably small, and $a$ is chosen appropriately, then the corresponding Dirichlet problem cannot have a solution.

Suppose in fact that a solution $u$ did exist corresponding to the given data $g$. Then by lemma 1, if $a$ is sufficiently small, say $a \leqslant a(\epsilon)$, we have

$$u \leqslant m + \epsilon \quad \text{on} \quad \Omega \cap \{r = a\} \tag{87}$$

where $m$ is the supremum of $g$ on the boundary set $\partial\Omega \cap \{r \geqslant a\}$. (The hypothesis $\mathscr{E}^* \leqslant 1 - \mu$ of lemma 1 is treated exactly as in the proof of theorem 1. We note also that if $m < 0$, then (87) must be replaced by $u \leqslant \epsilon$; the rest of the proof carries through unchanged, however.)

Since $g(P) > f(P)$, we have also

$$\mathscr{I} < \mathscr{C}_0(y, g, -\nu) \quad \text{at} \quad P$$

because $\mathscr{C}_0$ is increasing in its second argument. Consequently by lemma 3 there exists a constant $\epsilon_0$ such that the condition

$$g(P) - 5\epsilon' \leqslant m' \leqslant g(P) - \epsilon' \quad (m' = \sup u \quad \text{on} \quad \Omega \cap \{r = a\})$$

cannot occur when $\epsilon' \leqslant \epsilon_0$ and $a \leqslant a_0(\epsilon')$. Moreover, it is clear that we can choose $\epsilon_0$ and $a_0(\epsilon')$ to be independent of $\epsilon$.

Now suppose $\epsilon < \epsilon_0$ and let
$$a = \min\,(a(\epsilon), a_0(\epsilon), a_1(\epsilon))$$

where $a_1(\epsilon)$ is defined by the continuity condition

$$|f(y) - f(P)| \leqslant \epsilon \quad \text{if} \quad y \in \partial\Omega \cap \{r \leqslant a_1(\epsilon)\}.$$

We assert that the Dirichlet problem for $g$ cannot then have a solution.

Indeed, since $a \leqslant a(\epsilon)$ we have from (87) that $m' \leqslant m + \epsilon$. Thus using (86) and the fact that $f$ takes its maximum value at $P$, we obtain

$$m' \leqslant m + \epsilon \leqslant f(P) + 2\epsilon = g(P) - \epsilon. \tag{88}$$

On the other hand, by considering points $y$ in the set $\partial\Omega \cap \{r = a\}$, it is clear that

$$\begin{aligned} m' &\geqslant g(y) \geqslant f(y) - \epsilon \\ &\geqslant f(P) - 2\epsilon = g(P) - 5\epsilon, \end{aligned} \tag{89}$$

the first inequality of the second line following since $a \leqslant a_1(\epsilon)$. From (88) and (89),

$$g(P)-5\epsilon \leqslant m' \leqslant g(P)-\epsilon.$$

But this is impossible according to the earlier part of the proof, since $\epsilon$ is less than $\epsilon_0$ and $a \leqslant a_0(\epsilon)$. This completes the demonstration of theorem 2.

The following corollary generalizes theorem 1.

Corollary. *Suppose that equation* (1) *is singularly elliptic and that the asymptotic formulae* (26) *and* (27) *hold. Assume furthermore that* $\mathscr{B} \geqslant 0$ *for all sufficiently large values of* $u$.

*Let* $\Omega$ *be a bounded domain whose boundary is of class* $C^2$, *and suppose there is some point* $y$ *of the boundary surface where*

$$\mathscr{I} < \lim_{u\to\infty} \mathscr{C}_0(y,u,-\nu).$$

*Then there exists smooth boundary data for which no solution of the Dirichlet problem is possible.*

*Proof.* From the proof of theorem 2, one sees that its conclusion remains true even if (85) holds only at a single point $P$ on the boundary where $f$ takes its greatest value. Consequently, to prove the corollary, it is enough to start with sufficiently large *constant* boundary values $f$ and then to apply theorem 2.

While the global failure of the condition $\mathscr{I} \geqslant \mathscr{C}_0(y,f,-\nu)$ indicates that the Dirichlet problem is badly posed, it remains of interest to construct explicit boundary data such that

$$\mathscr{I} \geqslant \mathscr{C}_0(y,f,-\nu)-\epsilon$$

and yet for which no solution of the Dirichlet problem exists. We have the following particular result in this direction, complementing theorem 2.

Theorem 3. *Suppose that equation* (1) *is singularly elliptic and that the asymptotic formulae* (26) *and* (27) *hold. Assume furthermore that the function* $\mathscr{C}(x,u,p)$, $|p| \geqslant 1$, *is bounded below for* $u>0$ *and bounded above for* $u<0$.†

*Let* $\epsilon>0$ *be given. Then there exists a smoothly bounded domain* $\Omega$ *and smooth boundary data* $f$, *such that*

$$\mathscr{H} \geqslant -\mathscr{C}_0(y,f,\nu)-\epsilon \quad \text{and} \quad \mathscr{I} \geqslant \mathscr{C}_0(y,f,-\nu)-\epsilon \qquad (90)$$

*at each point* $y$ *of the boundary surface, yet the Dirichlet problem has no solution.*

*Proof.* Let $\bar{y}$ be a fixed point in the underlying Euclidean space, say the origin of coordinates for definiteness, and let $\bar{\nu}$ be a fixed direction at $\bar{y}$, say $\bar{\nu}=(1,0,\ldots,0)$. We can suppose without loss of generality that $\bar{y}$ and $\bar{\nu}$ satisfy the condition

$$\mathscr{C}_0(\bar{y},0,-\bar{\nu}) \geqslant -\mathscr{C}_0(\bar{y},0,\bar{\nu}),$$

since if the opposite inequality holds one can replace $u$ by $-u$.

Now put

$$k=\mathscr{C}_0(\bar{y},0,-\bar{\nu})-\tfrac{1}{2}\epsilon$$

and let $S$ denote the quadric surface.

$$x_1=\tfrac{1}{2}k(x_2^2+\ldots+x_n^2).$$

As in the proof of lemma 3, the domain to the right of $S$ will be denoted by $S^+$. We consider the set

$$\{r<a\} \cap S^+$$

† It is in any case bounded for $x$, $u$ in compact sets.

where $r$ denotes distance from $\bar{y}$, and we let $\Omega$ be the domain obtained from it by just barely rounding off the edges.

For this domain it is evident that

$$\mathscr{H} = \mathscr{I} = k \quad \text{at the boundary point } \bar{y}$$

and $$\mathscr{H} = \mathscr{I} = a^{-1} \quad \text{on the boundary set } \{r = a\}.$$

On the rounded-off portions of the boundary $\mathscr{H}$ and $\mathscr{I}$ can be assumed to be arbitrarily large, say greater than $2a^{-1}$ for definiteness. Next, we introduce smooth boundary data $f$ such that $f \equiv 0$ on the part of the boundary disjoint from $S$, $0 \leqslant f \leqslant 3\epsilon'$ on the part of the boundary coincident with $S$, and $f = 3\epsilon'$ at $\bar{y}$.

Clearly if $a$ and $\epsilon'$ are sufficiently small, then (90) holds at each point of $\partial\Omega$, while moreover

$$\mathscr{I}(\bar{y}) \leqslant \mathscr{C}_0(\bar{y}, 3\epsilon', -\bar{\nu}) - \tfrac{1}{2}\epsilon.$$

We now assert that if $\epsilon'$ and $a$ are still smaller if necessary, then the corresponding Dirichlet problem cannot have a solution. Suppose in fact that solution $u$ did exist for the domain $\Omega$ and the data $f$ described above. By hypothesis there exists a constant $R > 0$ such that

$$\mathscr{C}(x, u, p) \geqslant -1/R \quad \text{for} \quad |p| \geqslant 1, \quad u > 0.$$

Assuming $a \leqslant R$, theorem 6·3 then implies

$$u \leqslant 3 + 2R \quad \text{in} \quad \Omega, \quad (\epsilon' \leqslant 1).$$

Since $u$ is bounded above, the argument of lemma 3 may be applied. In particular, if $\epsilon'$ is sufficiently small and $a \leqslant a_0(\epsilon')$, then the condition

$$f(\bar{y}) - 5\epsilon' \leqslant m' \leqslant f(\bar{y}) - \epsilon' \tag{91}$$

cannot occur, where $m'$ denotes the supremum of $u$ on the part of the boundary of $\Omega$ which is disjoint from $S$. But with this final choice of $\epsilon'$ and $a$ the proof is completed, for by our construction

$$f(\bar{y}) = 3\epsilon' \quad \text{and} \quad m' = 0$$

and these values *do* satisfy (91).

*Comments*. The domain $\Omega$ constructed above is evidently convex if either $\mathscr{C}_0(\bar{y}, 0, -\bar{\nu})$ or $-\mathscr{C}_0(\bar{y}, 0, \bar{\nu})$ is positive, while otherwise it is necessarily non-convex.

In addition to the results noted at the end of § 16, Bernstein (1912, page 469) derived one further theorem for the case $n = 2$, $g = 2$, namely if $\mathscr{A}$ and $\mathscr{B}$ are analytic and have certain expansion properties for large $p$, and if (in our notation) the function $\mathscr{C}_0$ vanishes, then there are nonconvex domains for which the Dirichlet problem is not solvable. This result is clearly a special case of theorem 3.

## 19. The equation $\mathscr{A}D^2u = 0$

In this section we intend to specialize our earlier results to the important case when equation (1) has the form

$$\mathscr{A}(x, u, Du)\, D^2u = 0. \tag{1'}$$

This specialization is particularly interesting in view of the relatively complete state of the theory when $\mathscr{B} = 0$, as well as the simple nature of the conclusions which are obtained. We note in particular that the Euler–Lagrange equation associated with the regular variational problem

$$\delta \int F(Du)\, \mathrm{d}x = 0, \quad F \in C^3,$$

has the form (1′); in addition the representation (54) clearly holds, so that the following theorems are applicable in this case for all dimensions.

Theorems 1 and 2 below are concerned with the existence of solutions of the Dirichlet problem for arbitrarily given boundary data, while theorem 3 deals with the question of nonexistence.

THEOREM 1. *Let* $\Omega$ *be a bounded domain in n-dimensional Euclidean space, whose boundary is of class* $C^2$.

*Assume either* $n = 2$ *or that* (54) *holds. Suppose moreover that there exists a positive constant* $\mu$ *such that the invariant function* $\mathscr{E} = p\mathscr{A}p$ *satisfies the condition*

$$\mathscr{E} \geqslant \mu |p|$$

*for all sufficiently large values of* $p$.

*Then the Dirichlet problem for equation* (1′) *in* $\Omega$ *is solvable for arbitrarily given boundary data of class* $C^2$. *The solution is unique if* $\mathscr{A}$ *is independent of* $u$.

This is proved in exactly the same way as theorem 14·1, except that in obtaining an interior estimate for $|Dv|$ one can now apply either theorem 12·1 or 13·1. (Note that theorem 6·2 can be used at the first stage to obtain a bound for $|v|$ in $\Omega$.) Uniqueness follows from theorem 6·5.

THEOREM 2. *Let* $\Omega$ *be a bounded domain in n-dimensional Euclidean space, whose boundary is of class* $C^2$. *Assume that both representations* (26) *and* (54) *hold.*

*Then the Dirichlet problem for equation* (1′) *in* $\Omega$ *is solvable for arbitrarily given* $C^2$ *boundary values provided*

$$\mathscr{H}, \mathscr{I} > 0 \tag{92}$$

*at each point of the boundary surface, where the invariants* $\mathscr{H}$ *and* $\mathscr{I}$ *are defined in* § 9.

This is proved in exactly the same way as the first part of theorem 14·3, except that in obtaining an interior estimate for $|Dv|$ one can now apply either theorem 12·1 or 13·1.

The equality sign can be allowed in (92) provided also that

$$\sigma \mathscr{A}_0 \sigma = 0, \quad \mathscr{A} - \mathscr{A}_0 = o(\mathscr{E}/|p|),$$

and $\mathscr{E} \geqslant \mu$ for some positive constant $\mu$ and all sufficiently large values of $p$. Indeed, it is precisely these conditions which are needed to complete the proof of the second part of theorem 14·3.

THEOREM 2′. *Let* $\Omega$ *be a bounded strictly convex domain in n-dimensional Euclidean space, whose boundary is of class* $C^2$. *Assume either* $n = 2$ *or that the representation* (54) *holds.*

*Then the Dirichlet problem for equation* (1′) *in* $\Omega$ *is solvable for arbitrarily given boundary data of class* $C^2$.

This is proved in exactly the same way as the first part of theorem 14·5, except that the interior estimate for $|Dv|$ is obtained from either theorem 12·1 or 13·1.

Theorem 2′ holds also for convex (not necessarily strictly convex) domains $\Omega$ provided that the minimum eigenvalue of $\mathscr{A}$ is bounded below by a positive multiple of $|p|^{-2}$ as

$p$ becomes large. To see this, we note that $\mathscr{F}$ is then bounded below by a positive constant, whence the last part of the proof of theorem 14·5 is immediately applicable.

*Remarks*. The case $n = 2$ of theorem 2′ has been considered by a number of authors. The result is essentially due to Bernstein (1910, 1912), though in fact he never quite considered the situation in full generality. The most complete treatment is due to Nirenberg (1952). With the help of the *a priori* estimates of Ladyzhenskaya & Uraltseva (cf. § 1), the result for $n > 2$ and $\mathscr{A} = \mathscr{A}(p)$ can be obtained by exactly the hyperplane barrier methods employed by Bernstein and Nirenberg; cf. also the work of Gilbarg (1963) and Stampacchia (1963) for the case when (1′) has divergence structure or arises from a variational problem. The result of theorem 2′ seems to be new when $\mathscr{A}$ has the representation (54), as does also the result when $\Omega$ is convex but not strictly convex.

Turning to the question of non-existence of solutions, we have first the following immediate corollary of theorem 18·1.

THEOREM 3. *Suppose equation* (1′) *is singularly elliptic according to the definition given in* § 17. *Assume furthermore that the asymptotic formula* (26) *holds.*

*Let* $\Omega$ *be a bounded domain of class* $C^2$. *If either condition*

$$\mathscr{H} < 0 \quad \text{or} \quad \mathscr{I} < 0$$

*holds at a single point of the boundary surface, then there exists smooth boundary data having arbitrarily small absolute value for which no solution of the Dirichlet problem is possible.*

When $n = 2$ the asymptotic condition (26) can be dispensed with, and the conclusion made even sharper. The result is as follows.

THEOREM 3′. *Assume* $n = 2$. *Let equation* (1′) *be singularly elliptic, and suppose that*

$$\mathscr{E}^* \leqslant 1 - \mu$$

*for some positive constant* $\mu$ *and for all* $u$ *with* $|u| < 1$.

*If* $\Omega$ *is a bounded non-convex domain, then there exists smooth boundary data having arbitrarily small absolute value such that no solution of the Dirichlet problem is possible.*

*Proof*. According to a theorem of Léja & Wilkosz (cf. Valentine 1964, theorem 4·8) there must be some boundary point $P$ of $\Omega$ where there is an internally touching arc of a circle with negative curvature $k$. Now let $u$ be a solution of the Dirichlet problem in $\Omega$, with boundary values $f$, and let $m'$ be the supremum of $u$ on the set $\Omega \cap \{r = a\}$, where $r$ denotes distance from $P$. Then the following assertion holds:

*For any given* $\epsilon > 0$, *we have*

$$f(P) \leqslant m' + \epsilon$$

*for all sufficiently small values of* $a$ (*depending only on* $\epsilon$, $\Omega$, $k$, *and the structure of the equation*).

Assuming this for the moment, the proof can then be completed as in the case of theorem 18·1, with the above result taking the place of lemma 3.

To prove the assertion, we follow the argument used for the proof of lemma 3. The only slight changes to take into account are that the internally touching circular arc here replaces the quadric surface $S$, while instead of (82) we must use the relation

$$\mathscr{I} = \frac{\lambda \mathscr{A} \lambda}{1 - kd} k = (\lambda \mathscr{A} \lambda) k + O(a) = (1 - \nu \mathscr{A} \nu) k + O(a)$$

(since trace $\mathscr{A} = 1$). Now $\nu\mathscr{A}\nu = \mathscr{E}^* \leqslant 1-\mu$ by hypothesis, whence recalling that $k$ is negative we obtain

$$\mathscr{J} \leqslant \mu k + O(a).$$

The rest of the proof of lemma 3 carries over unchanged (and indeed in somewhat simpler form) to the present case. This completes the demonstration of theorem 3′. (A corresponding result can be given for higher dimensions, but involves the concept of boundary points where there is an internally touching spherical element with negative curvature. This and other similar generalizations are left to the reader.)

The preceding results can occasionally be combined in such a way as to produce necessary and sufficient conditions for the solvability of Dirichlet's problem. To illustrate this, we may state the following

THEOREM 4. *Let $\Omega$ be a bounded domain in n-dimensional Euclidean space, whose boundary is of class $C^2$. Assume either $n = 2$ or that the representation* (54) *holds.*

*Suppose further that* (1′) *is singularly elliptic, and that the asymptotic formulae* (26) *holds with the error term*

$$\mathscr{A} - \mathscr{A}_0 = o(\mathscr{E}/|p|).$$

*Assume finally that there exists a positive constant $\mu$ such that $\mathscr{E} \geqslant \mu$ for all sufficiently large values of $p$.*

*Then the Dirichlet problem for equation* (1′) *in $\Omega$ is solvable for arbitrarily given boundary data of class $C^2$ if and only if*

$$\mathscr{H},\quad \mathscr{J} \geqslant 0$$

*at each point of the boundary surface.*

This result is a combination of theorem 2 and theorem 3, with lemma 17·2 and the remark following the proof of theorem 2 being taken into account.

COROLLARY. *Let $\Omega$ be a bounded domain in n-dimensional Euclidean space, whose boundary is of class $C^2$. Assume either $n = 2$ or that* (54) *holds.*

*Suppose furthermore that* (1′) *has a well-defined genre $g$, $1 < g \leqslant 2$, and that the representation* (26) *holds with*

$$\mathscr{A} - \mathscr{A}_0 = o(|p|^{1-g}).$$

*Then the Dirichlet problem for equation* (1′) *in $\Omega$ is solvable for arbitrarily given boundary data of class $C^2$ if and only if $\mathscr{H}, \mathscr{J} \geqslant 0$ at each point of the boundary surface.*

The minimal surface equation obviously satisfies the hypotheses of this corollary, with $\mathscr{H} = \mathscr{J} = H$ in this case. Thus, as a special case, we obtain a recent result of Howard Jenkins and the author (1968) for the minimal surface equation in higher dimensions. In fact, the following more general result holds.

*The Dirichlet problem for the minimal surface equation in a domain of class $C^2$ is solvable for arbitrarily given continuous boundary values if and only if $H \geqslant 0$ at each point of the boundary. The solution is unique if it exists.*

*Proof.* The necessity of the condition $H \geqslant 0$ is obvious from what has gone before.

Now suppose that $H \geqslant 0$ at each point of the boundary of $\Omega$, and let continuous boundary values $f$ be given. To prove the existence of a corresponding solution of the Dirichlet problem, consider new boundary values $f_m$ of class $C^2$ such that

$$|f - f_m| \leqslant 1/m, \quad m = 1, 2, \ldots.$$

Each of the approximating problems has a unique solution $u_m$ according to the preceding corollary. Hence by the maximum principle (theorem 6·5) the sequence $\{u_m\}$ tends uniformly in $\overline{\Omega}$ to a limit function $u(x)$. Clearly $u = f$ on $\partial\Omega$; it remains only to show that $u$ is a solution of the minimal surface equation in $\Omega$.

To this end, we apply a recently announced result of E. De Giorgi, namely that the gradient of a solution of the minimal surface equation can be bounded in terms of the maximum of the solution and the distance to the boundary (cf. Bombieri, De Giorgi & Miranda 1969). Using this result and the Schauder theory, it follows that the functions $u_m$ are uniformly bounded in $C^3$ on any compact subset of $\Omega$. Consequently by Arzela's theorem and an easy argument we see that $u$ must be of class $C^2$ in $\Omega$ and that $u_m \to u$ in $C^2$ on any compact subset. Since the functions $u_m$ are solutions of the minimal surface equation in $\Omega$ it is clear that $u$ is also a solution in $\Omega$, completing the proof.

We note that the same technique applies when the boundary of $\Omega$ is not of class $C^2$ but can be uniformly approximated by $C^2$ boundaries which satisfy the condition $H \geqslant 0$ at each point.



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## Chapter IV

In this chapter we shall give an extended discussion of various examples and special cases in order to provide concrete illustrations of the general theory developed earlier. Sections 22 through 24 are devoted to the case of surfaces having prescribed mean curvature, where the results have special geometric significance beyond their simple interest as examples.

### 20. The minimal surface operator

Here we shall consider various equations arising when the left-hand side of (1) has the form

$$[(1+|Du|^2)\,I-DuDu]\,D^2u$$

as in the case of the minimal surface equation. We note that the coefficient matrix $\mathscr{A}$, after normalization to unit trace, is

$$\frac{(1+|p|^2)\,I-pp}{n+(n-1)\,|p|^2}$$

and that the corresponding invariants $\mathscr{H}$ and $\mathscr{I}$ are given by the simple relation (cf. §9)

$$\mathscr{H}=\mathscr{I}=H$$

where $H$ devotes the ordinary mean curvature of the boundary surface in question.

Perhaps the most important case concerns surfaces of constant mean curvature. As is well known, a smooth surface in $(n+1)$-dimensional Euclidean space with the non-parametric representation $z=u(x)$ has mean curvature $\Lambda$ provided that the function $u=u(x)$ satisfies

$$[(1+|Du|^2)\,I-DuDu]\,D^2u=n\Lambda(1+|Du|^2)^{\frac{3}{2}}.$$

The corresponding Dirichlet problem in a given domain $\Omega$ has the geometric content that the surface $z=u(x)$ is defined over $\Omega$ and takes on given ordinate values on the boundary set $\partial\Omega$. In particular, when $n=2$ the surface $z=u(x,y)$ spans a given simple closed curve in the three dimensional $(x, y, z)$-space.

As already remarked in the introduction of the paper, the following result holds:

*Let $\Omega$ be a bounded domain in n-dimensional Euclidean space, whose boundary is of class $C^2$. Then the Dirichlet problem in $\Omega$ for surfaces of constant mean curvature is solvable for arbitrarily given boundary data of class $C^2$, if and only if the mean curvature $H$ of the boundary surface $\partial\Omega$ satisfies the condition*

$$H\geqslant\frac{n}{n-1}|\Lambda| \tag{93}$$

*at each point of the boundary. The solution is unique if it exists.*

*Proof.* We first show that (93) is necessary. Suppose in fact that (93) fails at a single point. If $\Lambda\geqslant 0$, then theorem 18·1 shows that there exists boundary data having arbitrarily small absolute value for which no solution of the Dirichlet problem is possible. If $\Lambda<0$ the same proof applies, if we merely replace $u$ with $-u$.

The proof of sufficiency is more difficult. We shall give here an argument applicable only in two dimensions. The proof for higher dimensions will be deferred until §22, where in fact we shall derive an even stronger result.

Suppose then that $n = 2$. In this case (93) implies that $\Omega$ is contained in a ball of radius $1/2|\Lambda|$. The hypotheses of theorem 6·3 are then satisfied, and consequently the solvability of the Dirichlet problem is guaranteed by theorem 14·3. The uniqueness of solutions follows from theorem 6·5, whatever the dimension. This completes the demonstration of the theorem, subject of course to the deferred existence proof in higher dimensions.

As we have already remarked, when $n = 2$ condition (93) implies that $\Omega$ is convex and contained in a ball of radius $1/2|\Lambda|$. In higher dimensions no such simple conclusions are available, and (93) can in fact be satisfied even by domains which are not simply connected. For example, the torus generated by revolving the disk $(x-a)^2+y^2 = b^2$ about the $y$ axis satisfies the condition whenever $b < 1/3|\Lambda|$ and $a \geqslant 2b+3|\Lambda|b^2/(1-3|\Lambda|b)$.

The minimal surface operator also appears in the equation of Laplace and Gauss governing the free surface of a stationary fluid under the combined action of gravity and surface tension. The following result holds:

*Let $\Omega$ be a bounded domain in n-dimensional Euclidean space, whose boundary is of class $C^2$. Then the Dirichlet problem in $\Omega$ for the Gauss–Laplace equation*

$$[(1+|Du|^2)I-DuDu]\,D^2u = cu(1+|Du|^2)^{\frac{3}{2}}, \quad c > 0 \tag{94}$$

*is uniquely solvable for given $C^2$ boundary values $f$ provided that*

$$|f| \leqslant (n-1)\,H/c \tag{95}$$

*at each point of the boundary. Moreover, condition* (95) *is best possible.*

*Proof.* Existence follows directly from theorem 14·3 (with theorem 6·2 used at the final step.) By saying that (95) is best possible we mean that, given any domain $\Omega$ and any positive number $\epsilon$, then the following statements hold:

(A) If $H < 0$ at a single point of the boundary there exists smooth boundary data $g$ such that $|g| \leqslant \epsilon$ and yet for which no solution of the Dirichlet problem for (94) exists.

(B) If $H \geqslant 0$ at each point of the boundary there exists smooth boundary data $g$ such that $|g| \leqslant (n-1)\,H/c+\epsilon$ and yet for which no solution of the Dirichlet problem exists.

The first assertion follows by setting $f = 0$ and using theorem 18·2 (together with the fact that its conclusion remains true even if (85) holds only at a single point $P$ of the boundary where $f$ takes its greatest value).

In case (B), let $P$ be an arbitrary point on the boundary and let smooth boundary values $f$ be assigned such that
$$f = (n-1)\,H/c+\tfrac{1}{2}\epsilon \quad \text{at} \quad P$$
and
$$f \leqslant \min\{f(P), (n-1)\,H/c+\tfrac{1}{2}\epsilon\} \text{ elsewhere.}$$
The conclusion then follows exactly as in case (A), completing the proof of the theorem.

Corollary. *Let $\Omega$ be a ball of radius $R$. Then the Dirichlet problem above is solvable for arbitrarily given $C^2$ boundary values $f$ satisfying the condition $|f| \leqslant (n-1)/cR$. Conversely, if $|f| > (n-1)/cR$ at a single point of the boundary, then the Dirichlet problem is not well set.*

It would be interesting to know similarly whether the failure of condition (94) at a single point implies the existence of nearby data for which the Dirichlet problem cannot be solved.

Because of their general interest we shall consider two further equations,

$$[(1+|Du|^2)I-DuDu]\,D^2u = n\Lambda(1+|Du|^2)^{\frac{1}{2}\theta} \tag{96}$$

and

$$[(1+|Du|^2)I-DuDu]\,D^2u = cu(1+|Du|^2)^{\frac{1}{2}\theta} \tag{97}$$





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where $\Lambda$, $c$, and $\theta$ are constants, $c > 0$. The case $\theta = 3$ has already been discussed in the foregoing work. We may therefore restrict the following treatment to values $\theta \neq 3$.

When $\theta > 3$ it is evident that both equations are irregularly elliptic; accordingly the Dirichlet problem is not well posed in that case, whatever the domain.

When $\theta < 3$ a maximum principle is available for both equations (theorems 6·3 and 6·2), and $\mathscr{C}_0 = 0$. Also when $\theta \leqslant 2$ both equations are boundedly non-linear. Finally theorems 12·1 and 13·1 are applicable, with

$$\dot{\mathscr{D}} = 0, \qquad \dot{\mathscr{S}} = 0$$

and

$$\dot{\mathscr{D}} = c\,\frac{(1+|p|^2)^{\frac{1}{2}\theta-1}}{n-1} > 0, \quad \dot{\mathscr{S}} = c\,\frac{(1+|p|^2)^{\frac{1}{2}\theta-1}}{n} > 0,$$

respectively. The argument of theorem 14·3 therefore may be applied to prove the existence of solutions of the Dirichlet problem for (96) and (97) when $\theta < 3$.

On the other hand, according to theorem 18·1 the Dirichlet problem for $\theta < 3$ cannot generally be solved if $H < 0$ at some point of the boundary. A slight modification of the proof of the antecedent lemma 17·3 in fact allows this result to be strengthened: when $2 < \theta < 3$ the Dirichlet problem cannot always be solved if $H = 0$ at some point on the boundary (for this modification alone, we make the tacit assumption that the boundary of $\Omega$ is of class $C^3$ in the neighbourhood of points where $H = 0$). Combining the preceding results, we obtain the following theorem.

*Let $\Omega$ be a bounded domain in n dimensional Euclidean space, whose boundary is of class $C^2$. Then the Dirichlet problem for equations* (96) *and* (97) *in $\Omega$ is solvable for arbitrarily given $C^2$ boundary values under the following circumstances:*

(*i*) *when $\theta \leqslant 2$, if and only if $H \geqslant 0$ at each point of the boundary, and*
(*ii*) *when $2 < \theta < 3$, if and only if $H > 0$ at each point of the boundary.*

*When $\theta > 3$ the Dirichlet problem is not generally solvable, whatever the domain.*

In a short paper published in 1910 Bernstein listed both equations (96) and (97) as examples (though with $n$ equal to 2 and $\theta$ restricted to *integer* values), and showed that the corresponding Dirichlet problems are solvable for arbitrary (analytic) boundary data in an arbitrary strictly convex (analytic) domain if and only if $\theta \leqslant 2$.

In the same paper Bernstein introduced a further pair of equations, namely

$$(1+u_x^2)\,u_{xx} + 2u_x u_y u_{xy} + (1+u_y^2)\,u_{yy} = (1+u_x^2+u_y^2)^{\frac{1}{2}\theta}$$

and

$$(1+u_x^2)\,u_{xx} + 2u_x u_y u_{xy} + (1+u_y^2)\,u_{yy} = u(1+u_x^2+u_y^2)^{\frac{1}{2}\theta},$$

with the parameter $\theta$ once more being restricted to integer values. A recent note of mine (1967, page 1834) called attention to the $n$-dimensional generalizations of these equations, and gave precise conditions for the solvability of the corresponding Dirichlet problems. The interested reader should be able to supply the necessary justification of those results without undue difficulty.

## 21. Variational equations

Both the equation for surfaces having constant mean curvature and the equation for fluid interfaces under the action of gravity and surface tension can be obtained from an appropriate variational problem. It is the purpose of this section to present several further variational equations of interest, including higher dimensional generalizations of the problems introduced by Bernstein in his memoir on the calculus of variations (1912).

We begin with the example

$$\delta \int \sqrt{(1+|x|^2+|Du|^2)}\, \mathrm{d}x = 0,$$

for which the corresponding Euler–Lagrange equation has the form

$$[(1+|x|^2+|Du|^2)\, I - DuDu]\, D^2u = x\cdot Du.$$

By theorems 14·3 and 18·1 the Dirichlet problem in a bounded domain $\Omega$ with $C^2$ boundary is solvable for arbitrary boundary values of class $C^2$ if and only if $H \geqslant 0$ at each point of the boundary. By theorem 6·5 the solution is unique if it exists.

Consider next the problem

$$\delta \int \sqrt{(1+u^2+|Du|^2)}\, \mathrm{d}x = 0,$$

whose Euler–Lagrange equation has the form

$$[(1+u^2+|Du|^2)\, I - DuDu]\, D^2u = u(1+u^2+2|Du|^2).$$

Again by theorems 14·3 and 18·1 the Dirichlet problem is solvable for arbitrarily given boundary values if and only if $H \geqslant 0$ at each point of the boundary (note in particular that an *a priori* bound for $|u|$ is given by theorem 6·2). Uniqueness of the solution follows from theorem 6·6.

In both preceding cases the slight modification of the integrand from the area expression $\sqrt{(1+|Du|^2)}$ has no effect on the type of Dirichlet problem which can be solved. A rather different situation arises for the variational problem

$$\delta \int \phi(x) \sqrt{(1+|Du|^2)}\, \mathrm{d}x = 0$$

where $\phi(x)$ is a given positive function of class $C^2$. The corresponding Euler–Lagrange equation is

$$[(1+|Du|^2)\, I - DuDu]\, D^2u = -\frac{D\phi\cdot Du}{\phi}(1+|Du|^2);$$

thus while the left-hand side involves only the minimal surface operator, the right-hand side yields the relation

$$\mathscr{C}_0 = -\frac{1}{n-1}\,\frac{\sigma\cdot D\phi}{\phi}.$$

Hence the geometric conditions required for the solvability of Dirichlet's problem necessarily differ from those for the minimal surface equation. In particular, by theorem 14·3 the Dirichlet problem in a bounded domain $\Omega$ with $C^2$ boundary is solvable for arbitrary boundary values of class $C^2$ provided that

$$H \geqslant \frac{1}{n-1}\,\frac{\nu\cdot D\phi}{\phi} \tag{98}$$

at each point of the boundary. It can be shown that this condition is also necessary for the solvability of the Dirichlet problem for arbitrary data, though we shall not do so here.

The special case $\phi = \sqrt{(1+|x|^2)}$ was remarked by Bernstein (1912, page 479). If for definiteness we take $\Omega$ to be convex and to contain the origin, *then* (98) *is always satisfied.*





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When $\Omega$ is a ball of radius $R$ whose centre lies at a distance $S$ from the origin, (98) becomes

$$\frac{1}{R} \geqslant \frac{1}{n-1} \frac{S^2-R^2-|x|^2}{2R(1+|x|^2)},$$

that is
$$2(n-1)+(2n-1)|x|^2 \geqslant S^2-R^2.$$

The minimum value of $|x|$ encountered on the surface of the ball is $|R-S|$. Thus we obtain the following solvability condition for the Dirichlet problem in this special case,

$$(S-R)\left(\frac{n}{n-1}R-S\right) \leqslant 1;$$

in particular the problem is solvable whenever

(1) $\Omega$ contains the origin, or
(2) the radius of $\Omega$ is less than $2(n-1)$, or
(3) the centre of $\Omega$ lies in a ball of radius $2\surd\{n(n-1)\}$ about the origin.

This result serves to emphasize the delicacy of the situation when any but the simplest equations are treated. We observe specifically that even if the Dirichlet problem is solvable for a given ball $\Sigma$, it need not be solvable for all balls contained in $\Sigma$, nor even for all smaller concentric balls!

## 22. Surfaces having prescribed mean curvature

In this section we shall generalize the result of §20 to include cases where the surface $z = u(x)$ has a prescribed mean curvature $\Lambda(x)$ of class $C^1$.

An example will make clear that beyond the simple geometric condition (93), some additional requirements on the behaviour of $\Lambda(x)$ are necessary in order for the Dirichlet problem to be solvable. Consider in particular a smooth function $\Lambda(x)$ which is identically 2 when $|x| \leqslant \frac{1}{2}$ and vanishes identically when $|x| \geqslant 1$. Then (93) is satisfied if $\Omega$ is the unit ball, but according to a result of Bernstein (1910, page 243) no solution of the Dirichlet problem can possibly exist, no matter what boundary data is assigned.

In the following result we shall give a family of sufficient conditions, depending on a parameter $a$, such that the Dirichlet problem is solvable for a given domain $\Omega$ whenever $\Lambda(x)$ satisfies any one of the conditions in the family.

For points $y$ on the boundary of $\Omega$, let $k_1(y), \ldots, k_{n-1}(y)$ be the principal curvatures of the boundary surface, listed in decreasing order of size. Let $d = d(x)$ denote as usual the distance to the boundary of $\Omega$, and let $\Omega_1$ be the (open) subset of $\Omega$ consisting of points $x$ such that

(1) the closed ball of radius $d(x)$ centred at $x$ touches the boundary of $\Omega$ at a single point $y = y(x)$, and

(2) $d(x) < 1/k_1(y)$.

For $x$ in $\bar{\Omega}$ we can now define

$$H(x) = \begin{cases} \dfrac{1}{n-1}\sum \dfrac{k_i(y)}{1-k_i(y)\,d(x)}, & x \in \Omega_1 \cup \partial\Omega, \\ \infty, & x \notin \Omega_i \cup \partial\Omega. \end{cases}$$

Clearly $H(x)$ agrees with the ordinary mean curvature of the boundary surface for points $x$ on $\partial\Omega$. Moreover, although the fact is unnecessary for our present purposes, we remark that wnen the boundary is of class $C^3$ the function $H(x)$ can be interpreted as the mean curvature of the level surface $d =$ constant passing through the point $x$.

The maximum value of $d(x)$ in $\Omega$ will be called the internal radius of $\Omega$, and will be denoted by $d_1$. We can now state the main result of the section:

*Let $\Omega$ be a bounded domain in n-dimensional Euclidean space, whose boundary is of class $C^2$. Let $\Lambda(x)$ be of class $C^1$ in the closure of $\Omega$, and suppose that there is some constant a, $0 \leqslant a \leqslant 1/d_1$, such that*

$$\frac{n}{n-1}|\Lambda(x)| \leqslant (1-ad)\,H(x)+\frac{a}{n-1} \quad \textit{in} \quad \Omega. \tag{99}$$

*Then the Dirichlet problem in $\Omega$ for the equation*

$$[(1+|Du|^2)\,I-DuDu]\,D^2u = n\Lambda(x)\,(1+|Du|^2)^{\frac{3}{2}}$$

*is solvable for arbitrarily given $C^2$ boundary values if and only if*

$$H \geqslant \frac{n}{n-1}|\Lambda(y)|$$

*at each point y of the boundary. The solution is unique if it exists.*

The proof is somewhat different in the three cases $0 < a < 1/d_1$, $a = 1/d_1$, and $a = 0$, and accordingly we shall consider these situations separately.

*Case* 1. $0 < a < 1/d_1$

Supposing that the boundary of $\Omega$ is of class $C^3$, we assert that

$$|u| \leqslant \sup|f|+1/a \quad \text{in} \quad \Omega. \tag{100}$$

Assuming this for the moment, the result then follows exactly as in the proof of theorem 14·3, with (100) taking the place of the final hypothesis on the coefficient $\mathscr{B}$.

It is therefore sufficient to prove (100). Let

$$h(d) = \sqrt{\{2ad-(ad)^2\}}/a, \quad 0 \leqslant d < 1/a.$$

Since $0 \leqslant h < 1/a$ the assertion will follow provided we can establish the relation

$$|u(x)| \leqslant \sup|f|+h(d) \quad d = d(x), \tag{101}$$

for all $x$ in $\Omega$.

Suppose for contradiction that (101) fails at some point where $u > 0$. Since (101) necessarily holds on the boundary of $\Omega$, it follows that there exists a positive constant $\kappa$ and a point $\bar{x}$ in $\Omega$ such that

$$u(x) \leqslant \sup|f|+h(d)+\kappa \quad \text{in} \quad \Omega \tag{102}$$

with equality holding at $\bar{x}$. There are now two cases to consider, depending on whether $\bar{x}$ is in the set $\Omega_1$ or in its complement.

I. $\bar{x} \in \Omega_1$. According to the work of § 3, one sees easily that relations (8) hold and $d(x)$ is of class $C^2$ in $\Omega_1$. Hence, letting $v(x)$ denote the function on the right-hand side of (102), we find by direct computaion (as in the proof of lemma 4·1) that

$$\begin{aligned}\frac{[(1+|Dv|^2)\,I-DvDv]\,D^2v}{(1+|Dv|^2)^{\frac{3}{2}}} &= \frac{h''}{(1+h'^2)^{\frac{3}{2}}}-\frac{h'}{(1+h'^2)^{\frac{1}{2}}}\sum\frac{k_i(y)}{1-k_i(y)\,d(x)}\\ &= -a-(1-ad)\,(n-1)\,H(x) \leqslant n\Lambda(x)\end{aligned}$$

for $x$ in $\Omega_1$. With the help of a standard maximum principle argument this implies $u \equiv v$ in $\Omega_1$, which is impossible since $u < v$ on $\partial\Omega$.

II. $\bar{x} \notin \Omega_1$. Let $\bar{y}$ be any point where the closed ball of radius $d(\bar{x})$ centred at $\bar{x}$ touches the boundary of $\Omega$. Then for any point $x$ on the closed line segment joining $\bar{y}$ to $\bar{x}$ we have

$$d(x) = |x - \bar{y}|.$$

Since $h' > 0$, it follows from (102) that $Du \neq 0$ at $\bar{x}$. The level set $u =$ constant $= \bar{u}$ passing through $\bar{x}$ is consequently a surface of class $C^2$ in the neighbourhood of $\bar{x}$. This being the case, there exists a closed ball $\Sigma$ tangent to this surface at $\bar{x}$, such that $u \geqslant \bar{u}$ in $\Sigma$. But then by (102) it is clear that $d \geqslant d(\bar{x})$ in $\Sigma$, and *a fortiori* $d \geqslant d(\bar{x})$ on $\partial\Sigma$.

If the centre of $\Sigma$ is $x_0$ and its radius $\sigma_0$, then according to the geometric meaning of $d(x)$ the ball of radius $d(\bar{x}) + \sigma_0$ centred at $x_0$ must be contained in $\bar{\Omega}$. This implies first of all that $x_0$ is on the extension of the line segment joining $\bar{y}$ to $\bar{x}$, and in turn that $\bar{x}$ is in $\Omega_1$. This is contrary to assumption, and accordingly we have proved that (101) must hold whenever $u > 0$. Since a similar argument applies to points where $u < 0$, this completes the proof of (101).

*Case* 2. $a = 1/d_1$

Let $\bar{a}$ be an arbitrary constant, $1/2d_1 < \bar{a} < 1/d_1$. If $\Lambda$ satisfies (99) for $a = \bar{a}$, then by what has already been shown in case 1,

$$|u| \leqslant \sup |f| + 2d_1 \quad \text{in} \quad \Omega$$

(provided as always that the boundary of $\Omega$ is of class $C^3$). The solvability of the Dirichlet problem when $a = 1/d_1$ can now be established as in the proof of theorem 14·3, since each approximating problem obviously can be chosen so that (99) holds for some $a < 1/d_1$.

*Case* 3. $a = 0$

As in the proof of case 1, it is sufficient to establish an *a priori* estimate for $|u|$ when the boundary of $\Omega$ is of class $C^3$. To this end, we put

$$v(x) = \sup |f| + h(d), \quad 0 \leqslant d \leqslant d_1,$$

where
$$h(d) = \frac{e^{cd_1}}{c}(1 - e^{-cd}), \quad d = d(x),$$

and $c = n \sup |\Lambda(x)| + 1$. Now for $x \in \Omega_1$

$$\begin{aligned}
\frac{[(1+|Dv|^2)I - DvDv]D^2v}{(1+|Dv|^2)^{\frac{3}{2}}} &= \frac{h''}{(1+h'^2)^{\frac{3}{2}}} - \frac{h'}{(1+h'^2)^{\frac{1}{2}}}(n-1)H(x) \\
&\leqslant \frac{h''}{(1+h'^2)^{\frac{3}{2}}} + \frac{h'}{(1+h'^2)^{\frac{1}{2}}} n\Lambda(x) \\
&\leqslant n\Lambda(x) + \frac{h''}{(1+h'^2)^{\frac{3}{2}}} + c\left\{1 - \frac{h'}{(1+h'^2)^{\frac{1}{2}}}\right\} \\
&\leqslant n\Lambda(x) + \frac{h'' + ch'}{(1+h'^2)^{\frac{3}{2}}} = n\Lambda(x),
\end{aligned}$$

the second to last step holding by virtue of the fact that $h' \geqslant 1$. This being shown, the proof of case 1 carries over with only minor changes to the present situation. Thus we have

$$|u(x)| \leqslant \sup |f| + h(d) \leqslant \sup |f| + (e^{cd_1} - 1)/c$$

in $\Omega$. This completes the proof of case 3, and consequently also of the main result.

The necessity of the condition $H \geqslant (n/n-1)|\Lambda|$ follows directly from theorem 18·1 in case $\Lambda$ does not change sign. If $\Lambda$ takes on both positive and negative values the argument is somewhat more complicated but follows the same basic lines (we shall omit the details). Finally, the uniqueness of the solution is an immediate consequence of theorem 6·5.

The result of case 3 is worth stating as a separate corollary because of its special interest.

*Let $\Omega$ be a bounded domain in n-dimensional Euclidean space, whose boundary is of class $C^2$. Then the Dirichlet problem in $\Omega$ for surfaces having prescribed mean curvature $\Lambda(x)$ is solvable for arbitrarily given $C^2$ boundary values provided that*

$$|\Lambda(x)| \leqslant \frac{n-1}{n} H(x)$$

*at each point x in the closure of $\Omega$. The solution is unique if it exists.*

Since $H(x)$ obviously takes its minimum value on the boundary of $\Omega$, this corollary clearly includes as a special case the theorem stated in § 20. The results of § 20 are thus completely proved.

The case $a = 1/d_1$ of the main result is also of individual interest, and yields the sufficient condition:

$$H \geqslant \frac{n}{n-1}|\Lambda(y)|, \quad y \text{ on } \partial\Omega$$

and

$$|\Lambda(x)| \leqslant \frac{1}{nd_1} + \left(1-\frac{1}{n}\right)\left(1-\frac{d}{d_1}\right) H(x), \quad x \text{ in } \Omega.$$

By dropping the final term on the right-hand side we get the simple, though not especially strong, sufficient condition $|\Lambda(x)| \leqslant 1/nd_1$.

It is unfortunately not particularly easy to give an explicit formula for $H(x)$ for any but the simplest domains. On the other hand, for a given domain $\Omega$ it is always possible to employ in (99) the function $\tilde{H}(x)$ corresponding to any domain $\tilde{\Omega}$ containing $\Omega$. A case of particular importance occurs when $\tilde{\Omega}$ is a ball of radius $R$. If $r$ denotes distance from the centre of $\tilde{\Omega}$, then

$$d(x) = R-r, \quad \tilde{H}(x) = 1/r$$

and

$$(1-ad)\,\tilde{H}(x) + \frac{a}{n-1} = \frac{1-aR}{r} + \frac{an}{n-1}.$$

We have proved the following result:

*Let $\Omega$ be a domain of class $C^2$ in n-dimensional Euclidean space, contained in a ball of radius R. Let $\Lambda(x)$ be of class $C^1$ in the closure of $\Omega$, and suppose that there is some constant a, $0 \leqslant a \leqslant 1/R$, such that*

$$|\Lambda(x)| \leqslant a + \frac{n-1}{n}\,\frac{1-aR}{r} \quad \textit{in} \quad \Omega. \tag{99'}$$

*Then the Dirichlet problem in $\Omega$ for surfaces having prescribed mean curvature $\Lambda(x)$ is solvable for arbitrarily given boundary values of class $C^2$ if and only if*

$$H \geqslant \frac{n}{n-1}|\Lambda(y)|$$

*at each point y of the boundary. The solution is unique if it exists.*

It is interesting to observe that *condition* (99′) *is best possible*, in the sense that when $\Lambda(x)$ is subject only to the restrictions

$$|\Lambda(x)| \leqslant a + \frac{n-1}{n}\frac{1-aR}{r} + \epsilon \quad \text{in} \quad \Omega, \quad |\Lambda(x)| \leqslant \frac{n-1}{n} H \quad \text{on} \quad \partial\Omega$$

it may not be possible to solve the corresponding Dirichlet problem. To prove this, suppose that $\Omega$ is itself a ball of radius $R$, and consider a smooth non-negative function $\Lambda(x)$ such that

$$\Lambda(x) \leqslant \frac{n-1}{n}\frac{1}{R} \quad \text{when} \quad r = R$$

and
$$\Lambda(x) \equiv a + \frac{n-1}{n}\frac{1-aR}{r} + \epsilon \quad \text{when} \quad \delta < r < R - \delta$$

where $\delta$ will be determined later. Using the identity

$$\frac{[(1+|Du|^2)I - DuDu]D^2u}{(1+|Du|^2)^{\frac{3}{2}}} = \operatorname{div}\left(\frac{Du}{(1+|Du|^2)^{\frac{1}{2}}}\right)$$

we obtain, for any eventual solution $u$ of the Dirichlet problem in $\Omega$,

$$\int_\Omega n\Lambda(x)\,\mathrm{d}x \leqslant \oint_{\partial\Omega} \frac{|Du|}{(1+|Du|^2)^{\frac{1}{2}}}\,\mathrm{d}s \leqslant \omega_n R^{n-1},$$

where $\omega_n$ denotes the area of the unit sphere in $n$ dimensions. Also

$$\int_\Omega n\Lambda(x)\,\mathrm{d}x > \int_{\delta<r<R-\delta} (n-1)\left(\frac{1-aR}{r} + \frac{an}{n-1} + \epsilon\right)\mathrm{d}x$$
$$> \omega_n\left[\left(\frac{1}{R} + \epsilon\frac{n-1}{n}\right)(R-\delta)^n - \left(\frac{1}{\delta} + \epsilon\frac{n-1}{n}\right)\delta^n\right].$$

Combining the preceding two inequalities and letting $\delta \to 0$ yields

$$\omega_n R^{n-1} \geqslant \omega_n\left(1 + \epsilon R\frac{n-1}{n}\right)R^{n-1}$$

which is impossible. Hence a contradiction occurs for all sufficiently small values of $\delta$, proving our result.

The value of the preceding result lies in the fact that it does not require a specific calculation of the function $H(x)$ for the given domain $\Omega$. We conclude the section with another theorem having the same property.

*Let $\Omega$ be a bounded domain in n-dimensional Euclidean space, whose boundary is of class $C^2$. Let $\Lambda(x)$ be of class $C^1$ in the closure of $\Omega$, and suppose that*

$$|D\Lambda| \leqslant \frac{n}{n-1}\Lambda^2 \quad \textit{in} \quad \Omega. \tag{103}$$

*Then the Dirichlet problem in $\Omega$ for surfaces having prescribed mean curvature $\Lambda(x)$ is solvable for arbitrarily given $C^2$ boundary values if and only if*

$$H \geqslant \frac{n}{n-1}|\Lambda(y)| \tag{104}$$

*at each point y of the boundary. The solution is unique if it exists.*



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*Proof.* We shall show that (103) and (104) imply

$$|\Lambda(x)| \leqslant \frac{n-1}{n} H(x), \quad x \in \overline{\Omega}, \tag{105}$$

which will obviously complete the proof (see case 3 of the main result).

The rate of change of $H(x)$ with respect to $d$ in a direction normal to the boundary is clearly given by

$$\frac{\mathrm{d}H(x)}{\mathrm{d}d} = \frac{1}{n-1} \Sigma \left( \frac{k_i(y)}{1-k_i(y)\,d(x)} \right)^2, \quad x \in \Omega_1.$$

Consequently, letting a prime denote summation over all values of $i$ for which $k_i(y)$ is positive, we have

$$\frac{\mathrm{d}H(x)}{\mathrm{d}d} \geqslant \frac{1}{n-1} \Sigma' \left( \frac{k_i(y)}{1-k_i(y)\,d(x)} \right)^2$$

$$\geqslant \frac{1}{(n-1)^2} \left( \Sigma' \frac{k_i(y)}{1-k_i(y)\,d(x)} \right)^2 \geqslant H(x)^2$$

since $H(x) \geqslant 0$ in the present case because of (104). Thus for $x$ in $\Omega_1$

$$\frac{\mathrm{d}}{\mathrm{d}d} \left\{ \frac{n-1}{n} H(x) \right\} \geqslant \frac{n}{n-1} \left\{ \frac{n-1}{n} H(x) \right\}^2. \tag{106}$$

If we now integrate both (103) and (106) along normals to the boundary, and make use of (104) as an initial condition, it follows that

$$|\Lambda(x)| \leqslant \frac{n-1}{n} H(x), \quad x \in \Omega_1 \cup \partial\Omega.$$

Since (105) obviously holds at the remaining points of $\Omega$, this completes the proof.

## 23. Surfaces of prescribed mean curvature in central projection

Let $\Sigma$ be a smooth surface possessing a one-to-one central projection onto a domain $\Omega$ of the unit sphere in $(n+1)$ dimensions. If the radial values of $\Sigma$ are assigned on the boundary of $\Omega$, and if the mean curvature of $\Sigma$ is prescribed as a function over $\Omega$, we say that $\Sigma$ is a solution of the Dirichlet problem for surfaces of prescribed mean curvature (in central projection). We shall adopt the sign convention that the mean curvature of $\Sigma$ is based on the normal to the surface on the same side as the extended radius vector. The following result then holds.

*Let $\Omega$ be a domain on the unit sphere in $(n+1)$ dimensions, whose closure is contained in an open hemisphere and whose boundary is of class $C^2$. Let $\Lambda(x)$ be a non-negative function of class $C^1$ defined over the closure of $\Omega$. Then the Dirichlet problem in $\Omega$ for surfaces having prescribed mean curvature $\Lambda(x)$ (in central projection) is solvable for arbitrarily given radial values $f$ of class $C^2$ assigned on the boundary of $\Omega$, provided that*

$$H_g \geqslant \frac{n}{n-1} \Lambda(y) f(y)$$

*at each point $y$ of the boundary, where $H_g$ denotes the geodesic mean curvature of the boundary surface (with respect to the underlying unit sphere). The solution is unique if it exists.*

The assigned radial values $f$ are of course necessarily positive, so that (just as in the parallel projection problem studied in the previous section) the mean curvature of the boundary must be non-negative. In particular, when $n = 2$ the restriction of $\Omega$ to a hemisphere is already a consequence of the solvability condition.

If $\Lambda(x)$ is identically zero the solution is a minimal surface in central projection. For this case the solvability condition becomes simply

$$H_g \geqslant 0 \tag{107}$$

at each point of the boundary. The case $n = 2$ of this result is due to Radó (1932), his proof resting on the general solution of Plateau's problem for Jordan boundary curves. *We note that since the assigned boundary values do not appear in* (107), *this condition is also necessary.*

For non-constant curvatures it is remarkable that no restrictions need be placed on $\Lambda(x)$ beyond the simple condition that it does not take negative values. In this respect the Dirichlet problem in central projection is much simpler than the Dirichlet problem in parallel projection.

In order to prove the result we must first introduce an appropriate analytical structure for the problem. Let $\Sigma$ be a surface with one-to-one central projection onto the domain $\Omega$. If $P$ is a point of $\Sigma$ whose projection on the unit sphere is the point $x$, we write $v = v(x)$ for the radial distance of $P$ (that is, the distance from $P$ to the centre of the sphere). We next introduce a system of local coordinates over $\Omega$. To this end it can be supposed without loss of generality that $\Omega$ lies in the upper hemisphere of the unit sphere, and it is convenient to use stereographic coordinates based on projection from the South pole onto the tangent space at the North pole.

With these coordinates, we may in what follows consider $x$ to be a Cartesian $n$-tuple $(x_1, \ldots, x_n)$ in the tangent space, and we may similarly interpret $\Omega$ as a domain in this space.

LEMMA. *The surface $\Sigma$ has mean curvature $\Lambda(x)$ provided that $v = v(x)$ satisfies the equation*

$$[(a^2v^2+|Dv|^2)I-DvDv]D^2v = an\Lambda v(a^2v^2+|Dv|^2)^{\frac{3}{2}}$$
$$+\tfrac{1}{2}ax\cdot Dv\{(n-2)a^2v^2+(n-1)|Dv|^2\}+a^2v\{na^2v^2+(n+1)|Dv|^2\} \tag{108}$$

*for $x$ in $\Omega$, where $a = 4/(4+|x|^2)$. Moreover, we have*

$$H_g = \frac{H}{a}+\frac{x\cdot\nu}{2}$$

*where $H$ denotes the ordinary mean curvature of the boundary of $\Omega$ considered as a surface in the tangent space endowed with its Euclidean metric.*

This lemma is proved at the end of the section. Granting the result, in order to prove the first part of the theorem we must show that the Dirichlet problem for equation (108) in $\Omega$ has a positive solution $v$ for arbitrarily given positive boundary data $f$ of class $C^2$, provided that

$$H \geqslant a\left(\frac{n}{n-1}\Lambda(y)f-\tfrac{1}{2}y\cdot\nu\right) \tag{109}$$

at each point $y$ of the boundary.

The invariant $\mathscr{C}_0$ for (108) is given by

$$\mathscr{C}_0(x, v, \sigma) = a\left(\frac{n}{n-1}\Lambda(x)\,v+\tfrac{1}{2}\sigma\cdot x\right)$$

and $\partial\mathscr{C}_0/\partial v \geqslant 0$ since $\Lambda(x) \geqslant 0$. Using (109), it is therefore clear that the solvability conditions (62) in theorem 14·3 are satisfied. Consequently, were it not for the singularity in equation (108) when $v = 0$, theorem 14·3 could now be applied to obtain the existence of a solution of the given Dirichlet problem.

This being the case, it is natural to look for some artifice which avoids the fact that (108) is singular when $v = 0$. To this end, let $\kappa$ be a small positive constant, and consider the modified coefficient matrix

$$(a^2g(v)+|D^2v|)\,I-DvDv,$$

where

$$g(v) = \begin{cases} v^2 & \text{for} \quad |v| \geqslant \kappa, \\ \tfrac{1}{2}\kappa^2 & \text{for} \quad |v| \leqslant \tfrac{1}{2}\kappa \end{cases}$$

and $v^2 \leqslant g(v) \leqslant \kappa^2$ otherwise. Let (108) be written in the form $\mathscr{L}v = 0$, and let $\bar{\mathscr{L}}v = 0$ be the corresponding equation with modified coefficient matrix. By theorem 14·3 the Dirichlet problem for $\bar{\mathscr{L}}v = 0$ has a solution $v$ corresponding to the given boundary values $f$. We assert that $v$ is positive, and, moreover, that if $\kappa$ is chosen suitably small, then $v$ satisfies the given equation $\mathscr{L}v = 0$ as required.

To prove this, consider a family of comparison functions of the form

$$w(x) = h(r^2), \quad r = |x|,$$

where $h$, $h'$, $h'' > 0$. Clearly (since $x\cdot Dw \geqslant 0$ and $g(w) \geqslant w^2$)

$$\frac{\bar{\mathscr{L}}w}{a^2g(w)+|Dw|^2} \geqslant \Delta w - \frac{DwDwD^2w}{a^2g(w)+|Dw|^2} - an\Lambda w(a^2w^2+|Dw|^2)^{\frac{1}{2}} - \tfrac{1}{2}(n-1)\,ax\cdot Dw-(n+1)\,a^2w$$

$$\geqslant (n-1)\,(2-ar^2)\,h' - an\Lambda h(a^2h^2+4r^2h'^2)^{\frac{1}{2}}-(n+1)\,a^2h.$$

By assumption the closure of $\Omega$ lies in the open upper hemisphere of the unit sphere, and correspondingly we have $r < 2$ there. Thus there exists a constant $\epsilon > 0$ such that

$$ar^2 \leqslant 2-\epsilon \quad \text{in} \quad \Omega.$$

Hence $\bar{\mathscr{L}}w \geqslant 0$ in $\Omega$ if $\quad \epsilon(n-1)\,h' \geqslant (n+1)\,h+n\Lambda h(h^2+16h'^2)^{\frac{1}{2}}.$

This inequality, as well as the conditions $h$, $h'$, $h'' > 0$, is satisfied by the family of functions

$$h = \alpha\exp\left\{\frac{n+2}{\epsilon(n-1)}(r^2-4)\right\}, \quad 0 < \alpha \leqslant \kappa_1$$

where $\kappa_1$ is a suitably small constant depending only on $\epsilon$, $n$, and the supremum of $\Lambda$ in $\Omega$.

We can now prove the assertion. First, we have $v \geqslant 0$ according to theorem 6·2. Then, using the family of comparison functions above, we find by an easy application of the maximum principle of Eberhard Hopf (1927, page 152) that

$$v \geqslant \min\,(\kappa_1, \kappa_2)\,\exp\left\{-4\frac{n+2}{\epsilon(n-1)}\right\} \quad \text{in} \quad \Omega$$

where $\kappa_2$ $(> 0)$ is the infimum of the given boundary values. Choosing $\kappa$ to be the constant on the right-hand side of the last inequality completes the proof of the assertion. Hence we have established the existence of a positive solution of (108) taking on any given positive twice continuously differentiable boundary values $f$.

Uniqueness of the solution is essentially a consequence of a theorem of Alexandrov (1956). In fact, a stronger result holds: if $\Sigma$ and $\overline{\Sigma}$ both possess one-to-one central projections onto a domain $\Omega$, if the radial values $f$ of $\Sigma$ and $\bar{f}$ of $\overline{\Sigma}$ on the boundary of $\Omega$ satisfy $f \leqslant \bar{f}$, and if the mean curvatures $\Lambda(x)$ and $\overline{\Lambda}(x)$ finally satisfy the condition $\Lambda(x) \geqslant \overline{\Lambda}(x)$, then the radial values of $\Sigma$ never exceed the radial values of $\overline{\Sigma}$. For completeness we include a proof of this result which is independent of the work of Alexandrov (though it uses the same basic idea).

Let $v(x)$ and $\bar{v}(x)$ be the radial values in question, and suppose for contradiction that $v(x)$ exceeds $\bar{v}(x)$ at some point $x$ in $\Omega$. Then there exists a constant $\bar{\alpha} > 1$ and a point $\bar{x}$ in $\Omega$ such that

$$v(x) \leqslant \bar{\alpha}\bar{v}(x) \quad \text{in} \quad \Omega, \qquad v(\bar{x}) = \bar{\alpha}\bar{v}(\bar{x}).$$

Let $u(x) = \bar{\alpha}\bar{v}(x)$. Using the fact that $\bar{v}$ is a solution of (108), we find that

$$[(a^2u^2+|Du|^2)I-DuDu]\,D^2u = an(\overline{\Lambda}/\bar{\alpha})\,u\,(a^2u^2+|Du|^2)^{\frac{3}{2}} + \tfrac{1}{2}ax\cdot Du\{(n-2)\,a^2u^2+(n-1)\,|Du|^2\}+a^2u\{na^2u^2+(n+1)\,|Du|^2\}.$$

Since $\overline{\Lambda}/\bar{\alpha} \leqslant \Lambda$ we conclude from the maximum principle of Eberhard Hopf that $u \equiv v$ in $\Omega$, which contradicts the given boundary conditions. This completes the proof of the theorem, subject of course to the deferred lemma.

*Proof of the lemma*

It is well known (and easily verified using rectangular Cartesian coordinates) that the local mean curvature $\Lambda$ of a surface satisfies the variational equation

$$\delta(\mathrm{d}A)+n\Lambda\,\delta(\mathrm{d}V) = 0,$$

variations in the volume $V$ being considered positive when they occur in the direction of the normal defining $\Lambda$.

For a surface $\Sigma$ defined by the function $v(x)$ as in the lemma, we have

$$\mathrm{d}V = \frac{1}{n+1}v^{n+1}\,\mathrm{d}\omega(x) = \frac{a^n}{n+1}v^{n+1}\,\mathrm{d}x$$

where $\mathrm{d}\omega$ is the element of area on the unit sphere, and $a = 4/(4+|x|^2)$; see figure 2. In the same way, using the fact that stereographic projection is conformal, it is not hard to obtain

$$\mathrm{d}A = v^n\surd\{1+|Dv|^2/a^2v^2\}\,\mathrm{d}\omega(x) = (av)^{n-1}\surd\{a^2v^2+|Dv|^2\}\,\mathrm{d}x.$$

If we put

$$F(x,v,p) = (av)^{n-1}\surd\{a^2v^2+|p|^2\}, \quad G(x,v) = \frac{na^n}{n+1}v^{n+1}$$

it follows from variational calculus that

$$\mathrm{div}\,\{F_p(x,v,Dv)\} = F_v+\Lambda G_v$$

which upon expansion yields equation (108).

To prove the second part of the lemma, note that since stereographic projection is conformal the relation between $H$ and $H_g$ must necessarily be linear,

$$H_g = \alpha H+\beta\cdot\nu.$$

To determine the coefficients we can thus resort to special cases where the boundary of $\Omega$ is a plane or a sphere.

Considering first the boundaries $x \cdot \nu = 0$ we find that $\beta$ must be directed along a ray from the origin, that is $\beta = \gamma x$ where $\gamma$ is a scalar. Taking next the case $x \cdot \nu =$ constant and evaluating at the point $x = r\nu$, we get

$$H = 0, \quad H_g = \frac{\sin \frac{1}{2}\theta}{\cos \frac{1}{2}\theta} = \tfrac{1}{2}r, \quad \beta \cdot \nu = \gamma r$$

whence $\gamma = \frac{1}{2}$. Finally, putting $|x| = r = R$, we find at any point on this sphere

$$H = 1/r, \quad H_g = \frac{\cos \theta}{\sin \theta} = \frac{4 - r^2}{4r}, \quad \beta \cdot \nu = -\tfrac{1}{2}r$$

so that $\alpha = \frac{1}{4}(4 + r^2)$, completing the proof of the lemma.

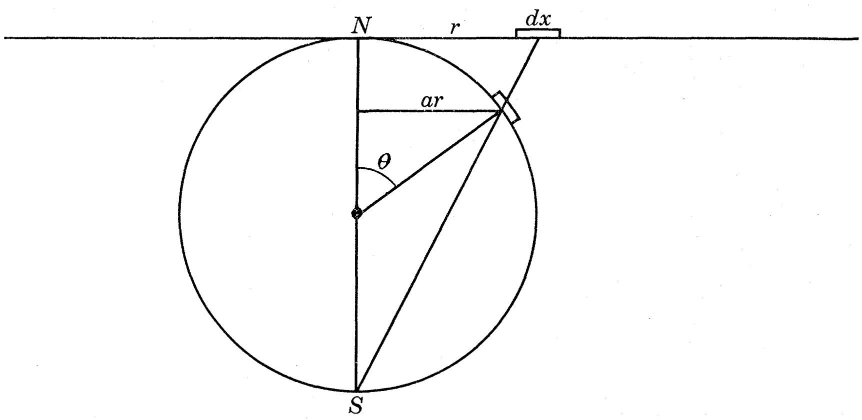


FIGURE 2

*Remark.* The above results, as well as those of the preceding section, can be generalized to the case where the mean curvature $\Lambda$ is allowed to depend on $v$ and $Dv$ in addition to $x$. We shall consider these matters in more detail in another paper to appear later.

## 24. THE DIFFERENTIAL EQUATION OF CONSTANT MEAN CURVATURE

In 1760 Lagrange published a new method—essentially the variational technique still used—for determining the differential equation to be satisfied by functions which minimize an integral expression under appropriate boundary conditions and side conditions (Euler's earlier method had been based on geometrical considerations and was moreover not particularly suited to problems involving functions of more than one independent variable). As an example illustrating the advantages of his general procedure Lagrange considered the problem of minimizing the area integral

$$\iint \sqrt{\{1 + u_x^2 + u_y^2\}}\, dx\, dy$$

of a surface $z = u(x, y)$ while keeping the volume integral

$$\iint u \, dx \, dy$$

fixed. He was correspondingly led to the differential equation

$$\frac{\partial}{\partial x}\left(\frac{u_x}{\sqrt{\{1+u_x^2+u_y^2\}}}\right)+\frac{\partial}{\partial y}\left(\frac{u_y}{\sqrt{\{1+u_x^2+u_y^2\}}}\right) = \text{constant} = 2\Lambda \tag{110}$$

which upon expansion yields the well-known equation for surfaces of constant mean curvature. This is the first partial differential equation to have been obtained by the methods of the calculus of variations, and is justly famous on this account as well as for its obvious geometric interest.

In the final part of his memoir, Lagrange noted that spheres (or more precisely their upper and lower hemispheres) generate a four-parameter family of exact solutions of equation (110), but beyond this he did not pursue the problem from the point of view either of the calculus of variations or of the differential equation itself. Further analysis of Lagrange's equation awaited the later interpretation of $\Lambda$ as the mean curvature of the surface $z = u(x, y)$. This interpretation in particular established that the upper and lower parts of right circular cylinders form a new family of exact solutions.

While the minimal surface equation ($\Lambda = 0$) shortly became the object of serious investigation, owing to the transformation formulae initiated by Monge and Legendre, no corresponding analytical techniques were found for dealing with the case of non-zero curvature. Accordingly it was not until 1841 that a further family of exact solutions was discovered by Delaunay. Delaunay proposed to determine the surface of revolution having constant mean curvature, and found that the generating curve must be such as would be traced out by the focus of a conic section, if the conic section itself rolls without slipping along the axis in question.† The particular case of a rolling parabola accounts for the catenary generating the minimal surface of revolution. Although no further exact solutions have been discovered since Delaunay's surfaces of revolution, a result of Bonnet (1853) should be noted for its related interest. If we consider a surface $S$ of constant mean curvature $\Lambda$, where $\Lambda$ may be taken positive without loss of generality, then the two parallel surfaces at distances $1/2\Lambda$ and $1/\Lambda$ in the direction of the preferred normal vector to $S$ have, respectively, constant Gaussian curvature $4\Lambda^2$ and constant negative mean curvature $-\Lambda$. A proof can be given on the basis of the formulae in § 4. In fact, one easily establishes that the mean and Gaussian curvatures $\bar{\Lambda}$ and $\bar{K}$ of the parallel surfaces at distance $d$ from $S$ are given by

$$\bar{\Lambda} = \frac{\Lambda - Kd}{1-2\Lambda d+Kd^2}, \quad \bar{K} = \frac{K}{1-2\Lambda d+Kd^2},$$

where $K$ is the Gaussian curvature of $S$, and the result follows at once.‡

The fact that there are few known exact solutions is of course neither unduly surprising nor seriously disturbing, though it naturally indicates the importance of general theoretical investigations. From the beginning of the eighteenth century mathematicians were aware

† Euler had found the differential equations corresponding to Delaunay's solutions as early as 1732, though he did not attempt their integration. The generating curves are illustrated by Minkowski (1906, page 582).

‡ See also W. Blaschke, *Differentialgeometrie*, Bd. I, p. 119. Berlin: Springer (1930).

of the necessity to formulate and solve problems in the calculus of variations corresponding to definite boundary conditions. In fact, already in 1785 Meusnier had clearly stated the Plateau problem: among all surfaces passing through a given closed curve not lying in a plane, to find that one whose area is least. Similarly, an early Dirichlet type problem for equation (110) was proposed by Gergonne (1816): Find the surface of minimum area among all those which are bounded by the sides of a square, and which include between themselves and the square a given volume. In view of the difficulty of such problems it is not to be wondered at that few attempted their study until well into this century.

In a series of remarkable papers appearing between the years 1907 and 1912 Bernstein reopened the Dirichlet problem for elliptic partial differential equations of the type (1) in two independent variables. While these papers were concerned for the most part with a fairly general situation, the example of surfaces of constant mean curvature still figures prominently and was undoubtedly an important stimulus to Bernstein's thought. At the beginning of his work Bernstein evidently considered equation (110) as unexceptional, for he states (1907, page 1027) that the corresponding Dirichlet problem in a circular domain is solvable for arbitrarily given continuous boundary data. By 1910, however, he had realized the various difficulties in the problem—in particular, that the inhomogeneous term on the right-hand side grows too rapidly in $|Du|$ for such a conclusion to be true.

At this time (1910, page 243) he obtained the remarkable theorem that if a domain $\Omega$ properly contains a circle of radius $1/|\Lambda|$ then no solution of (110) can possibly exist over $\Omega$, irrespective of the boundary data which might be assigned.† On page 244 Bernstein suggested a procedure for determining when a solution exists, and applied this to the particular case of an elliptical domain with zero boundary data. He writes furthermore: 'Il serait, sans doute, intéressant d'indiquer la marche systématique à suivre pour poivoir décider dans chaque cas particulier, par un nombre fini d'essais, si le problème est possible, ou non. Je n'ais pas l'intention de m'occupier ici de cette question.' Apart from the result discussed in the following paragraph, in fact, Bernstein never returned to this question.

Equation (110) appears again in a note in the *Comptes Rendus, Paris* (1910, page 515) where it is stated (among other things) that the Dirichlet problem is not generally solvable for convex domains. This result is of course a consequence of the theorem noted in the previous paragraph; Bernstein's general argument, however, is directed to the case when the domain may be arbitrarily small as well as convex, as is apparent from the later published proof of the result (1912, pages 465–469). This work is of considerable interest for the non-solvability of Dirichlet's problem, even though it is not sufficiently delicate to make a distinction between those convex domains for which a solution of Dirichlet's problem for equation (110) is always possible and those for which it is not (cf. § 20). Bernstein did not in fact realize that such a distinction might exist, for in a footnote to the theorem on page 465 he remarks: 'Il n'est pas douteux, d'ailleurs, qu'en complétent convenablement la démonstration on trouveriat que tous les contours jouissent de la méme propriété.'

A similar situation occurs in his treatment (1912, page 479) of the variational problem

$$\delta \int \surd(1+|x|^2)\,\surd(1+|Du|^2)\,\mathrm{d}x = 0, \quad n = 2.$$

† Other proofs and generalizations of this fact have been given by Heinz (1955), Chern (1965), and Finn (1965).



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He states there that this extremum problem does not, in general, admit a solution even for convex domains (compare § 21 of the present work). The adjoining remark is prophetic: 'Ce probléme, qui diffère si peu du problème de Plateau, rend compte, il me semble, des difficultés qu'on a du recontrer en cherchant des démonstrations directes de l'existence d'une surface minima pasant par un contour donné'. The example which Bernstein notes at the foot of page 483 provides still another instance where a more delicate treatment is required (see § 20).

To conclude the discussion of Bernstein's work, attention should be drawn to the final paragraphs of the memoir of 1912. That the result stated there, 'Si le problème de Dirichlet est toujours possible pour toute fonction analytique sur un contour convex, il est également possible si la fonction est supposée seulement continue sur le contour', cannot be correct follows from an example of Finn (1954, page 399). The problem is of course that of obtaining interior estimates for the gradient of a solution in terms of a maximum bound, but independent of the particular boundary data. While such estimates are available for certain equations in two independent variables (Finn 1963; Serrin 1963) and for most uniformly elliptic equations (cf. Ladyzhenskaya & Uraltseva 1964), there are scarcely any non-uniformly elliptic equations in higher dimensions for which such conclusions are known (in this regard, E. De Giorgi has recently shown the existence of such an interior estimate for the minimal surface equation in higher dimensions; see § 19).

During the period between Bernstein's memoir of 1912 and Leray's fundamental paper of 1939, I am not aware of any contributions to the problem of Dirichlet for surfaces of constant mean curvature. We have already commented in the introduction on the relation between Leray's work and Bernstein's. It does not seem to have been noticed before, however, that Leray's results can be applied to the Dirichlet problem for equation (110), provided a certain degree of care is taken. In particular, the following result can be obtained.

*Let* $\Omega$ *be a bounded domain in two-dimensional Euclidean space, whose boundary is of class* $C^5$. *Then the Dirichlet problem for equation* (110) *in* $\Omega$ *is solvable for arbitrarily given boundary values of class* $C^5$ *provided that* $\kappa \geqslant 2|\Lambda|$ *at each point of the boundary, where* $\kappa$ *denotes the curvature of the boundary curve.*

To see this, one may use corollary II on page 282 of Leray's paper (a straightforward but somewhat tedious calculation verifies the hypotheses) together with the argument given in § 20 of the present paper.

In a review article which appeared in 1961, Akhiezer & Petrovsky attribute to Bernstein the result that the Dirichlet problem for surfaces of constant mean curvature is solvable in any region which is entirely inside a circle of radius $1/|\Lambda|$. This statement seems to have resulted from reading more into some of Bernstein's remarks than was actually intended.

The question of uniqueness of surfaces having prescribed mean curvature is included in the important work of Alexandrov (1956–58) on surfaces in the large defined by functions of their principal curvatures. In addition to considering various Dirichlet-type problems in parallel and central projection, Alexandrov also obtained general results concerning the uniqueness of compact surfaces. These latter results imply in particular that a compact non-self-intersecting surface of constant mean curvature is a sphere, generalizing earlier work of Liebmann (1900) who required that the surface be convex. While this result applies equally in any number of dimensions, a complementary theorem of Hopf (1951) states that a simply connected compact surface of constant mean curvature in three dimensions is a sphere. The literature on these and related questions is fairly extensive, and the interested reader is referred to Alexandrov's paper for further details.

In another recent paper, Finn (1965) has obtained the following result: *Let* $u = u(x, y)$ *be a solution of* (110) *defined over the circular domain* $x^2+y^2 < 1/\Lambda^2$. *Then*

$$u \equiv \text{const.} + \sqrt{\{(1/\Lambda^2)-x^2-y^2\}}.$$

This is the natural analogue for equation (110) of Bernstein's celebrated theorem that a minimal surface $z = u(x, y)$ defined over the entire plane must itself be a plane. There are finally some conclusions of a differential geometric character concerning surfaces of constant mean curvature (e.g. the lines of curvature on such a surface form an isothermal net, etc.) for which we refer the reader to standard treaties on classical differential geometry or the theory of surfaces.

The discussion would not be complete without mention of the important series of papers of Heinz (1954) and Werner (1957, 1960) concerning the existence of surfaces of constant mean curvature $\Lambda$ in three-dimensional Euclidean space with arbitrary Jordan curves as assigned boundary. The surfaces in question are defined parametrically, and consequently (as in the problem of Plateau for minimal surfaces) may have self-intersections, multiple coverings, and branch points at which the local representation is singular. These matters aside, the elegant sufficient condition for solvability due to Werner is that the Jordan curves in question should be contained in a ball of radius $R$ where $1/R > 2|\Lambda|$. It is remarkable that this condition almost exactly parallels the solvability condition $\kappa \geqslant 2|\Lambda|$ for the Dirichlet problem for arbitrary assigned data.

This work has been in progress over a number of years and accordingly I have discussed it with a great many of my friends. To all of these, and particularly to D. E. Edmunds and J. C. C. Nitsche, I express my heartiest thanks and gratitude. The manuscript was written at the University of Sussex during a sabbatical leave of absence from the University of Minnesota. I am most grateful to both these institutions for the opportunity which they provided.



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*This paper is dedicated*
*to the memory of*
*Sergei Natanovich Bernstein*
*1880–1968*



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